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On Finding Exact Solutions to Coupled Systems of Partial Differential Equations by the NDM

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Abstract In this article, we develop a new method called the Natural Decomposition Method (NDM). We use the NDM to find exact solutions for coupled systems of linear and nonlinear partial differential equations. The proposed method is a combination of the Natural Transform Method (NTM) and Adomian Decomposition method (ADM). Besides, the NDM avoid round-off errors which leads to solutions in closed form. The new method always lead to an exact or approximate solution in the form of rapidly convergence series. Hence, the Natural Decomposition Method is elegant refinement of the existing methods and can easily be used to solve a wide class of Linear and Nonlinear Partial Differential Equations.

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1. Introduction

Linear and Nonlinear coupled systems of partial differential equations plays a significant role in applied sciences. They are used to describe shallow water wave equations, wave propagation equations and to examine the chemical diffusion model of Brusselator equation. Thus, many researchers become interested in developing various method for solving coupled systems of partial differential equations, such as; Adomian Decomposition Method (ADM) [1], Variational Iteration Method (VIM) [2], Laplace Decomposition Method (LDM) [3, 4], Modified Laplace Decomposition Method (MLDM) [5], and the Reduced Differential Transform Method (RDTM) [6–8].

In this article, we develop a new computational algorithm for solving Linear and Nonlinear coupled systems of partial differential equations called the Natural Decomposition Method (NDM). The new method is a combination of the well-known technique Adomian Decomposition Method (ADM) [1] and the new integral transform in this century which is called the Natural transform Method (NTM) [9]. The proposed method always lead to exact or approximate solution in the form of a rapidly convergence series with elegant

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computational terms. This shows the efficiency, flexibility and applicability of the new method. Various applications are solved to show the reliability and accuracy of the new approach. Hence the Natural Decomposition Method is a powerful mathematical tool for solving coupled systems of partial differential equations and it is a refinement of the existing methods.

In this paper, we solve the following coupled systems of partial differential equations: First, we consider the coupled system of linear partial differential equation of the form:

$$\begin{aligned} v_t(x,t) + w_x(x,t) &= 0 \\ w_t(x,t) + v_x(x,t) &= 0, \end{aligned} \tag{1.1}$$

subject to the initial conditions

$$\begin{aligned} v(x,0) &= e^x \\ w(x,0) &= e^{-x}. \end{aligned} \tag{1.2}$$

Secondly, we consider the non-homogeneous linear coupled system of partial differential equation of the form:

$$\begin{aligned} v_t(x,t) - w_x(x,t) - (v(x,t) - w(x,t)) &= -2 \\ w_t(x,t) + v_x(x,t) - (v(x,t) - w(x,t)) &= -2, \end{aligned} \tag{1.3}$$

subject to the initial conditions

$$\begin{aligned} v(x,0) &= e^x + 1 \\ w(x,0) &= e^x - 1. \end{aligned} \tag{1.4}$$

Finally, we consider coupled system of nonlinear partial differential equation of the form:

$$\begin{aligned} v_t(x,t) + w(x,t)v_x(x,t) + v(x,t) &= 1 \\ w_t(x,t) - v(x,t)w_x(x,t) - w(x,t) &= 1, \end{aligned} \tag{1.5}$$

subject to the initial conditions

$$\begin{aligned} v(x,0) &= e^x \\ w(x,0) &= e^{-x}. \end{aligned} \tag{1.6}$$

The remaining structure of this paper is organized as follows: In Sections 2 and 3, the NDM and ADM are introduced. In Section 4, we give the methodology of the NDM. Section 5 is devoted to three applications of the NDM to show the effectiveness, simplicity and reliability of the new method. Section 6 contains discussion and conclusion of this paper.

2. Background Materials

In this section, we present some background about the nature of the Natural Transform Method (NTM). Assume we have a function $f(t)$, $t \in (-\infty, \infty)$, and then the general integral transform is defined as follows [10, 11]:

$$\Im\left[f(t)\right](s) = \int_{-\infty}^{\infty} K(s,t)\, f(t)\, dt, \tag{2.1}$$

where $K(s,t)$ represent the kernel of the transform, s is the real (complex) number which is independent of t. Note that when $K(s,t)$ is e^{-st}, $t\, J_n(st)$ and $t^{s-1}(st)$, then Eq. (2.1) gives, respectively, Laplace transform, Hankel transform and Mellin transform.

Now, for $f(t),\ t \in (-\infty, \infty)$ consider the integral transforms defined by:

$$\Im\left[f(t)\right](u) = \int_{-\infty}^{\infty} K(t)\, f(ut)\, dt, \tag{2.2}$$

and

$$\Im\left[f(t)\right](s,u) = \int_{-\infty}^{\infty} K(s,t)\, f(ut)\, dt. \tag{2.3}$$

It is worth mentioning here when $K(t) = e^{-t}$, Eq. (2.2) gives the integral Sumudu transform, where the parameter s replaced by u. Moreover, for any value of n the generalized Laplace and Sumudu transform are respectively defined by [10] and [11]:

$$\ell\left[f(t)\right] = F(s) = s^n \int_0^{\infty} e^{-s^{n+1}t}\, f(s^n t)\, dt, \tag{2.4}$$

and

$$\mathbb{S}\left[f(t)\right] = G(u) = u^n \int_0^{\infty} e^{-u^n t}\, f(tu^{n+1})\, dt. \tag{2.5}$$

Note that when $n = 0$, Eq. (2.4) and Eq. (2.5) are the Laplace and Sumudu transform respectively.

3. Definitions and Properties of the N-Transform

The N–Transform of the function $f(t)$ for $t \in (-\infty, \infty)$ is defined by [10] and [11]:

$$\mathbb{N}\left[f(t)\right] = R(s,u) = \int_{-\infty}^{\infty} e^{-st}\, f(ut)\, dt; \quad s,\, u \in (-\infty, \infty), \tag{3.1}$$

where $\mathbb{N}\left[f(t)\right]$ is the natural transformation of the time function $f(t)$ and the variables s and u are the natural transform variables. Note that Eq. (3.1) can be written in the form [10, 11]:

$$\begin{aligned}
\mathbb{N}\left[f(t)\right] &= \textstyle\int_{-\infty}^{\infty} e^{-st}\, f(ut)\, dt; \quad s,\, u \in (-\infty, \infty) \\
&= \left[\textstyle\int_{-\infty}^{0} e^{-st}\, f(ut)\, dt; \quad s,\, u \in (-\infty, 0)\right] + \left[\textstyle\int_{0}^{\infty} e^{-st}\, f(ut)\, dt; \quad s,\, u \in (0, \infty)\right] \\
&= \mathbb{N}^{-}\left[f(t)\right] + \mathbb{N}^{+}\left[f(t)\right] \\
&= \mathbb{N}\left[f(t)H(-t)\right] + \mathbb{N}\left[f(t)H(t)\right] \\
&= R^{-}(s,u) + R^{+}(s,u),
\end{aligned}$$

where $H(.)$ is the Heaviside function.

It is worth mentioning here, if the function $f(t)H(t)$ is defined on the positive real axis, $t \in (0, \infty)$ and in the set

$$A = \left\{ f(t) : \exists\, M,\, \tau_1,\, \tau_2 > 0,\ |f(t)| < M e^{\frac{|t|}{\tau_j}},\ if\ t \in (-1)^j \times [0, \infty) \right\},$$

then we define the Natural transform (N–Transform) as:

$$\mathbb{N}\left[f(t)H(t)\right] = \mathbb{N}^{+}\left[f(t)\right] = R^{+}(s,u) = \int_{0}^{\infty} e^{-st} f(ut)\, dt; \quad s,\, u \in (0,\infty)\,. \tag{3.2}$$

Note if $u = 1$, then Eq. (3.2) can be reduced to the Laplace transform and if $s = 1$, then Eq. (3.2) can be reduced to the Sumudu transform.

Now we give some of the N–Transforms and the conversion to Sumudu and Laplace.

Table 1. Special N–Transforms and the conversion to Sumudu and Laplace.

$f(t)$	$\mathbb{N}\left[f(t)\right]$	$\mathbb{S}\left[f(t)\right]$	$\ell\left[f(t)\right]$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
$\frac{t^{n-1}}{(n-1)!}, n{=}1, 2, ...$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$
$\sin(t)$	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{1+s^2}$

Remark 3.1. The reader can read more about the N–Transform in [12] and [6].

Some basic Theorems of the N–Transforms are given as follows [6, 10–12]:

Theorem 3.2. *If $R(s,u)$ is the Natural transform and $F(s)$ is the Laplace transform of the function $f(t) \in A$, then* $\mathbb{N}^{+}\left[f(t)\right] = R(s,u) = \frac{1}{u}\int_0^{\infty} e^{-\frac{st}{u}} f(t)\, dt = \frac{1}{u}F\left(\frac{s}{u}\right)$.

Theorem 3.3. *If $R(s,u)$ is the Natural transform and $G(u)$ is the Sumudu transform of the function $f(t) \in A$, then* $\mathbb{N}^{+}\left[f(t)\right] = R(s,u) = \frac{1}{s}\int_0^{\infty} e^{-t} f\left(\frac{ut}{s}\right)\, dt = \frac{1}{s}G\left(\frac{u}{s}\right)$.

Theorem 3.4. *If* $\mathbb{N}^{+}\left[f(t)\right] = R(s,u)$, *then* $\mathbb{N}^{+}\left[f(at)\right] = \frac{1}{a}R(s,u)$.

Theorem 3.5. *If* $\mathbb{N}^{+}\left[f(t)\right] = R(s,u)$, *then* $\mathbb{N}^{+}\left[f'(t)\right] = \frac{s}{u}R(s,u) - \frac{f(0)}{u}$.

Theorem 3.6. *If* $\mathbb{N}^{+}\left[f(t)\right] = R(s,u)$, *then* $\mathbb{N}^{+}\left[f''(t)\right] = \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f'(0)}{u}$.

Remark 3.7. The proofs of the above theorems can be found in [10, 11].

Remark 3.8. The Natural transform is a linear operator. That is, if a and b are non–zero constants, then $\mathbb{N}^{+}\left[a\, f(t) \pm b\, g(t)\right] = a\mathbb{N}^{+}\left[f(t)\right] \pm b\mathbb{N}^{+}\left[g(t)\right] = aF^{+}(s,u) \pm bG^{+}(s,u)$. Moreover, $F^{+}(s,u)$ and $G^{+}(s,u)$ are the N–Transforms of $f(t)$ and $g(t)$, respectively.

4. Survey of the Natural Decomposition Method

We illustrate the Natural Decomposition Method (NDM) algorithm by considering the general nonlinear non-homogeneous system of PDEs of the form:

$$\begin{aligned} v_t(x,t) + w_x(x,t) + F_1\ (v,w) &= g_1(x,t) \\ w_t(x,t) + v_x(x,t) + F_2\ (v,w) &= g_2(x,t), \end{aligned} \tag{4.1}$$

subject to the initial conditions

$$\begin{aligned} v(x,0) &= h_1(x) \\ w(x,0) &= h_2(x), \end{aligned} \tag{4.2}$$

where $F_1(v,w)$ and $F_2(v,w)$ are the nonlinear operators and $g_1(x,t)$ and $g_2(x,t)$ are the non-homogeneous terms (source terms).

We apply the N–Transform to Eq. (4.1) and Eq. (4.2) to get:

$$\begin{aligned} &\tfrac{s}{u}v(x,s,u) - \tfrac{v(x,0)}{u} + \mathbb{N}^+\left[\, w_x(x,t)\right] + \mathbb{N}^+\left[\, F_1(v,w)\right] = \mathbb{N}^+\left[g_1(x,t)\right] \\ &\tfrac{s}{u}w(x,s,u) - \tfrac{w(x,0)}{u} + \mathbb{N}^+\left[\, v_x(x,t)\right] + \mathbb{N}^+\left[\, F_2(v,w)\right] = \mathbb{N}^+\left[g_2(x,t)\right]. \end{aligned} \tag{4.3}$$

By substituting the given initials conditions in Eq. (4.2) into Eq. (4.3) we obtain:

$$\begin{aligned} v(x,s,u) &= \tfrac{h_1(x)}{s} + \tfrac{u}{s}\mathbb{N}^+\left[\, g_1(x,t)\right] - \tfrac{u}{s}\mathbb{N}^+\left[\, w_x(x,t) + F_1(v,w)\right] \\ w(x,s,u) &= \tfrac{h_2(x)}{s} + \tfrac{u}{s}\mathbb{N}^+\left[\, g_2(x,t)\right] - \tfrac{u}{s}\mathbb{N}^+\left[\, v_x(x,t) + F_2(v,w)\right]. \end{aligned} \tag{4.4}$$

Taking the inverse N–Transform of Eq. (4.4), we obtain

$$\begin{aligned} v(x,t) &= G_1(x,t) - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\, w_x(x,t) + F_1(v,w)\right]\right] \\ w(x,t) &= G_2(x,t) - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\, v_x(x,t) + F_2(v,w)\right]\right], \end{aligned} \tag{4.5}$$

where $G_1(x,t)$ and $G_2(x,t)$ are the terms coming from the source terms.

Then we assume an infinite series solutions for the unknown functions $v(x,t)$ and $w(x,t)$ are given in the form:

$$\begin{aligned} v(x,t) &= \textstyle\sum_{n=0}^{\infty} v_n(x,t) \\ w(x,t) &= \textstyle\sum_{n=0}^{\infty} w_n(x,t). \end{aligned} \tag{4.6}$$

Moreover, the nonlinear terms $F_1(v,w)$ and $F_2(v,w)$ can easily be decomposed as follows:

$$\begin{aligned} F_1(v,w) &= \textstyle\sum_{n=0}^{\infty} A_n(x,t) \\ F_2(v,w) &= \textstyle\sum_{n=0}^{\infty} B_n(x,t), \end{aligned} \tag{4.7}$$

where A_n and B_n are the Adomian polynomials and can easily be computed with following formulas:

$$\begin{aligned} A_n &= \tfrac{1}{n!}\tfrac{d^n}{d\lambda^n}\left[F\left(\textstyle\sum_{j=0}^{n} \lambda^j v_j(x,t)\right)\right]_{\lambda=0} \\ B_n &= \tfrac{1}{n!}\tfrac{d^n}{d\lambda^n}\left[F\left(\textstyle\sum_{j=0}^{n} \lambda^j w_j(x,t)\right)\right]_{\lambda=0}, \end{aligned} \tag{4.8}$$

where $n = 0, 1, 2, \ldots$.

By substituting Eq. (4.8) into Eq. (4.7) we obtain:

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} v_n(x,t) &= G_1(x,t) - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\textstyle\sum_{n=0}^{\infty} w_{nx}(x,t) + \sum_{n=0}^{\infty} A_n\right]\right] \\ \textstyle\sum_{n=0}^{\infty} w_n(x,t) &= G_2(x,t) - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\textstyle\sum_{n=0}^{\infty} v_{nx}(x,t) + \sum_{n=0}^{\infty} B_n\right]\right]. \end{aligned} \tag{4.9}$$

Therefore, from Eq. (4.9) above, we can easily generate the recursive relation as follows:

$$v_0(x,t) = G_1(x,t)$$

$$v_1(x,t) = -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\sum_{n=0}^{\infty} w_{0x}(x,t) + \sum_{n=0}^{\infty} A_0\right]\right]$$

$$v_2(x,t) = -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\sum_{n=0}^{\infty} w_{1x}(x,t) + \sum_{n=0}^{\infty} A_1\right]\right].$$

Thus,

$$v_{n+1}(x,t) = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[w_{nx}(x,t) + A_n\right]\right], \qquad n \geq 0. \tag{4.10}$$

Similarly,

$$w_0(x,t) = G_2(x,t)$$

$$w_1(x,t) = -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\sum_{n=0}^{\infty} v_{0x}(x,t) + \sum_{n=0}^{\infty} B_0\right]\right]$$

$$w_2(x,t) = -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\sum_{n=0}^{\infty} v_{1x}(x,t) + \sum_{n=0}^{\infty} B_1\right]\right].$$

Eventually, we obtain:

$$w_{n+1}(x,t) = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[v_{nx}(x,t) + B_n\right]\right], \qquad n \geq 0. \tag{4.11}$$

Hence, the exact or approximate solutions of the nonlinear systems are given by:

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t)$$

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t).$$

5. Worked Examples

In this section, we apply the NDM to two coupled systems and then compare our solutions to existing exact solutions.

Example 5.1. Consider the linear coupled system of partial differential equation of the form:

$$\begin{aligned} v_t(x,t) + w_x(x,t) &= 0 \\ w_t(x,t) + v_x(x,t) &= 0, \end{aligned} \tag{5.1}$$

subject to the initial conditions

$$\begin{aligned} v(x,0) &= e^x \\ w(x,0) &= e^{-x}. \end{aligned} \tag{5.2}$$

By taking the Natural transform on both sides of Eq. (5.1), we obtain:

$$\begin{aligned} &\tfrac{sv(x,s,u)}{u} - \tfrac{v(x,0)}{u} + \mathbb{N}^+\left[w_x(x,t)\right] = 0 \\ &\tfrac{sw(x,s,u)}{u} - \tfrac{w(x,0)}{u} + \mathbb{N}^+\left[v_x(x,t)\right] = 0. \end{aligned} \tag{5.3}$$

By substituting the given initial conditions of Eq. (5.2) into Eq. (5.3), we obtain:

$$\begin{aligned} v(x,s,u) &= \tfrac{e^x}{s} - \tfrac{u}{s}\mathbb{N}^+\left[w_x(x,t)\right] \\ w(x,s,u) &= \tfrac{e^{-x}}{s} - \tfrac{u}{s}\mathbb{N}^+\left[v_x(x,t)\right]. \end{aligned} \tag{5.4}$$

Taking the inverse Natural transform of Eq. (5.4), we obtain:

$$\begin{aligned} v(x,t) &= e^{x} - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[w_x(x,t)\right]\right] \\ w(x,t) &= e^{-x} - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[v_x(x,t)\right]\right]. \end{aligned} \tag{5.5}$$

We now assume an infinite series solution of the unknown function $v(x,t)$ and $w(x,t)$ of the form:

$$\begin{aligned} v(x,t) &= \textstyle\sum_{n=0}^{\infty} v_n(x,t) \\ w(x,t) &= \textstyle\sum_{n=0}^{\infty} w_n(x,t). \end{aligned} \tag{5.6}$$

Then by using Eq. (5.5), we can re-write Eq. (5.6) in the form:

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} v_n(x,t) &= e^{x} - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[\textstyle\sum_{n=0}^{\infty} w_n(x,t)\right]\right] \\ \textstyle\sum_{n=0}^{\infty} w_n(x,t) &= e^{-x} - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[\textstyle\sum_{n=0}^{\infty} v_n(x,t)\right]\right]. \end{aligned} \tag{5.7}$$

Then from Eq. (5.7), we can easily build the general recursive relation as follows:

$$\begin{aligned} v_0(x,t) &= e^{x} \\ v_1(x,t) &= -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[w_0(x,t)\right]\right] \\ v_2(x,t) &= -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[w_1(x,t)\right]\right] \\ v_3(x,t) &= -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[w_2(x,t)\right]\right]. \end{aligned} \tag{5.8}$$

Then,

$$v_{n+1} = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[w_n(x,t)\right]\right], \qquad n \geq 0. \tag{5.9}$$

Similarly,

$$\begin{aligned} w_0(x,t) &= e^{-x} \\ w_1(x,t) &= -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[v_0(x,t)\right]\right] \\ w_2(x,t) &= -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[v_1(x,t)\right]\right] \\ w_3(x,t) &= -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[v_2(x,t)\right]\right]. \end{aligned} \tag{5.10}$$

Thus,

$$w_{n+1} = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_n(x,t)\right]\right], \qquad n \geq 0. \tag{5.11}$$

Therefore, by using the general recursive relation derived in Eq. (5.9) and (5.11), we can easily compute the remaining values of the unknown functions $v(x,t)$ and $w(x,t)$ as follows:

$$\begin{aligned} v_1(x,t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[w_0(x,t)\right]\right] \\ &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[-e^{-x}\right]\right] \\ &= e^{-x}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[1\right]\right] \\ &= e^{-x}\mathbb{N}^{-1}\left[\frac{u}{s^2}\right] \\ &= te^{-x}. \end{aligned} \tag{5.12}$$

$$\begin{aligned} w_1(x,t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_1(x,t)\right]\right] \\ &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[e^{x}\right]\right] \\ &= -e^{x}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[1\right]\right] \\ &= e^{x}\mathbb{N}^{-1}\left[\frac{u}{s^2}\right] \\ &= -te^{x}. \end{aligned} \tag{5.13}$$

$$\begin{aligned} v_2(x,t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[w_1(x,t)\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[-te^{x}\right]\right] \\ &= e^{-x}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t\right]\right] \\ &= e^{x}\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right] \\ &= \frac{t^2e^{x}}{2!}. \end{aligned} \tag{5.14}$$

$$\begin{aligned} w_2(x,t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_1(x,t)\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[-te^{-x}\right]\right] \\ &= e^{-x}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t\right]\right] \\ &= e^{-x}\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right] \\ &= \frac{t^2e^{-x}}{2!}. \end{aligned} \tag{5.15}$$

$$\begin{aligned}
v_3(x,t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[w_2(x,t)\right]\right] \\
&= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[\frac{-t^2e^{-x}}{2!}\right]\right] \\
&= \frac{e^{x}}{2!}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t^2\right]\right] \\
&= \frac{e^{-x}}{2!}\mathbb{N}^{-1}\left[\frac{2u^3}{s^4}\right] \\
&= \frac{t^3e^{-x}}{3!}.
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
w_3(x,t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_2(x,t)\right]\right] \\
&= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[\frac{t^2e^{x}}{2!}\right]\right] \\
&= \frac{-e^{x}}{2!}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t^2\right]\right] \\
&= \frac{-e^{x}}{2!}\mathbb{N}^{-1}\left[\frac{2u^3}{s^4}\right] \\
&= \frac{-t^3e^{x}}{3!}.
\end{aligned}$$

Hence, the approximate series solutions of $v(x,t)$ and $w(x,t)$ are given by:

$$\begin{aligned}
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \\
&= v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots \\
&= e^{x} - te^{-x} + \frac{t^2e^{x}}{2!} + \frac{t^3e^{-x}}{3!} + \cdots \\
&= e^{x}\left(1 + \frac{t^2}{2!} + \frac{t^4}{2!} + \cdots\right) + e^{-x}\left(t + \frac{t^3}{3!} + \frac{t^5}{3!} + \cdots\right) \\
&= e^{x}\cosh(t) + e^{-x}\sinh(t).
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
w(x,t) &= \sum_{n=0}^{\infty} w_n(x,t) \\
&= w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + \cdots \\
&= e^{x} - te^{x} + \frac{t^2e^{-x}}{2!} - \frac{t^3e^{x}}{3!} + \cdots \\
&= e^{-x}\left(1 + \frac{t^2}{2!} + \frac{t^4}{2!} + \cdots\right) - e^{x}\left(t + \frac{t^3}{3!} + \frac{t^5}{3!} + \cdots\right) \\
&= e^{x}\cosh(t) - e^{x}\sinh(t).
\end{aligned}$$

Therefore, the exact solutions of Eq. (5.1) are given by:

$$v(x,t) = e^{x}\cosh(t) + e^{-x}\sinh(t)$$

$$w(x,t) = e^{x}\cosh(t) - e^{x}\sinh(t).$$

The exact solutions are in agreement with the results obtained by ADM in [1].

Example 5.2. We next consider the coupled system of linear nonhomogeneous partial differential equation of the form:

$$\begin{aligned} &v_t(x,t) - w_x(x,t) - (v(x,t) - w(x,t)) = -2 \\ &w_t(x,t) + v_x(x,t) - (v(x,t) - w(x,t)) = -2, \end{aligned} \tag{5.18}$$

subject to the initial conditions

$$\begin{aligned} &v(x,0) = e^{x} + 1 \\ &w(x,0) = e^{x} - 1. \end{aligned} \tag{5.19}$$

Taking the Natural Transform of derivatives on both sides of Eq. (5.18), we obtain:

$$\begin{aligned} &\tfrac{s}{u}v(x,s,u) - \tfrac{1}{u}v(x,0) - \mathbb{N}^{+}\left[w_x(x,t)\right] - \mathbb{N}^{+}\left[v(x,t)\right] + \mathbb{N}^{+}\left[w(x,t)\right] = \tfrac{-2}{s} \\ &\tfrac{s}{u}w(x,s,u) + \tfrac{1}{u}v(x,0) - \mathbb{N}^{+}\left[v_x(x,t)\right] - \mathbb{N}^{+}\left[v(x,t)\right] + \mathbb{N}^{+}\left[w(x,t)\right] = \tfrac{-2}{s}. \end{aligned} \tag{5.20}$$

Then by substituting the given initial conditions of Eq. (5.19) into Eq. (5.20), we have:

$$\begin{aligned} &v(x,s,u) = \tfrac{1+e^{x}}{s} - \tfrac{2u}{s^2} + \tfrac{u}{s}\mathbb{N}^{+}\left[w_x(x,t) + v(x,t) - w(x,t)\right] \\ &w(x,s,u) = \tfrac{-1+e^{x}}{s} - \tfrac{2u}{s^2} + \tfrac{u}{s}\mathbb{N}^{+}\left[v(x,t) - w(x,t) - v_x(x,t)\right]. \end{aligned} \tag{5.21}$$

By taking the inverse Natural Transform of Eq. (5.21), we obtain:

$$\begin{aligned} &v(x,t) = e^{x} + 1 - 2t + \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[w_x(x,t) + v(x,t) - w(x,t)\right]\right] \\ &w(x,t) = e^{x} - 1 - 2t + \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[v(x,t) - w(x,t) - v_x(x,t)\right]\right]. \end{aligned} \tag{5.22}$$

Now, we assume an infinite series solutions of the unknown functions $v(x,t)$ and $w(x,t)$ of the form:

$$\begin{aligned} &v(x,t) = \textstyle\sum_{n=0}^{\infty} v_n(x,t) \\ &w(x,t) = \textstyle\sum_{n=0}^{\infty} w_n(x,t). \end{aligned} \tag{5.23}$$

Then by using Eq. (5.22) we can easily re-write Eq. (5.23) in the form:

$$\begin{aligned} &\textstyle\sum_{n=0}^{\infty} v_n(x,t) = e^{x} + 1 - 2t + \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} w_{nx}(x,t) + \sum_{n=0}^{\infty} v_n(x,t) - \sum_{n=0}^{\infty} w_n(x,t)\right]\right] \\ &\textstyle\sum_{n=0}^{\infty} w_n(x,t) = e^{x} - 1 - 2t + \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} v_n(x,t) - \sum_{n=0}^{\infty} w_n(x,t) - \sum_{n=0}^{\infty} v_{nx}(x,t)\right]\right]. \end{aligned} \tag{5.24}$$

Therefore, from Eq. (5.24), we can easily generate the recursive relation as follows:

$$v_0(x,t) = 1 + e^x - 2t$$

$$v_1(x,t) = \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[w_{0x}(x,t) + v_0(x,t) - w_0(x,t)\right]\right]$$

$$v_2(x,t) = \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[w_{1x}(x,t) + v_1(x,t) - w_1(x,t)\right]\right]$$

$$v_3(x,t) = \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[w_{2x}(x,t) + v_2(x,t) - w_2(x,t)\right]\right].$$

If we continue in the same manner, we will eventually have:

$$v_{n+1}(x,t) = \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[w_{nx}(x,t) + v_n(x,t) - w_n(x,t)\right]\right], \qquad n \geq 0. \tag{5.25}$$

Similarly, for the unknown function $w(x,t)$ we have:

$$w_0(x,t) = -1 + e^x - 2t$$

$$w_1(x,t) = \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[v_0(x,t) - w_0(x,t) - v_{0x}(x,t)\right]\right]$$

$$w_2(x,t) = \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[v_1(x,t) - w_1(x,t) - v_{1x}(x,t)\right]\right]$$

$$w_3(x,t) = \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[v_2(x,t) - w_2(x,t) - v_{2x}(x,t)\right]\right].$$

Then the general recursive relation of the unknown function $w(x,t)$ is given by:

$$w_{n+1}(x,t) = \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[v_n(x,t) - w_n(x,t) - v_{nx}(x,t)\right]\right], \qquad n \geq 0. \tag{5.26}$$

Hence, from the recursive relation derived in Eq. (5.2) and Eq. (5.26) we can easily compute the remaining components of the unknown functions $v(x,t)$ and $w(x,t)$ as follows:

$$\begin{aligned} v_1(x,t) &= \mathbb{N}^+\left[\frac{u}{s}\mathbb{N}^+\left[w_{0x}(x,t) + v_0(x,t) - w_0(x,t)\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[e^x + 2\right]\right] \\ &= \left(e^x + 2\right)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+[1]\right] \\ &= \left(e^x + 2\right)\mathbb{N}^{-1}\left[\frac{u}{s^2}\right] \\ &= te^x + 2t. \end{aligned}$$

$$\begin{aligned} w_1(x,t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[v_0(x,t) - w_0(x,t) - v_{0x}(x,t)\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[-e^x + 2\right]\right] \\ &= \left(-e^x + 2\right)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+[1]\right] \\ &= \left(-e^x + 2\right)\mathbb{N}^{-1}\left[\frac{u}{s^2}\right] \\ &= 2t - te^x. \end{aligned}$$

$$\begin{aligned}
v_2(x,t) &= \mathbb{N}^{+}\left[\frac{u}{s}\mathbb{N}^{+}\left[w_{1x}(x,t)+v_1(x,t)-w_1(x,t)\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[te^{x}\right]\right] \\
&= \left(e^{x}\right)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t\right]\right] \\
&= e^{x}\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right] \\
&= \frac{t^2}{2!}e^{x}.
\end{aligned}$$

$$\begin{aligned}
w_2(x,t) &= \mathbb{N}^{+}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_1(x,t)-w_1(x,t)-v_{1x}(x,t)\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[te^{x}\right]\right] \\
&= e^{x}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t\right]\right] \\
&= e^{x}\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right] \\
&= \frac{t^2}{2!}e^{x}.
\end{aligned}$$

$$\begin{aligned}
v_3(x,t) &= \mathbb{N}^{+}\left[\frac{u}{s}\mathbb{N}^{+}\left[w_{2x}(x,t)+v_2(x,t)-w_2(x,t)\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[\frac{t^2}{2!}e^{x}\right]\right] \\
&= \frac{1}{2!}e^{x}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t^2\right]\right] \\
&= \frac{1}{2!}e^{x}\mathbb{N}^{-1}\left[\frac{2u^3}{s^4}\right] \\
&= \frac{t^3}{3!}e^{x}.
\end{aligned}$$

$$\begin{aligned}
w_3(x,t) &= \mathbb{N}^{+}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_2(x,t)-w_2(x,t)-v_{2x}(x,t)\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[-\frac{t^2}{2!}e^{x}\right]\right] \\
&= -\frac{1}{2!}e^{x}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t^2\right]\right] \\
&= -\frac{1}{2!}e^{x}\mathbb{N}^{-1}\left[\frac{2u^3}{s^4}\right] \\
&= -\frac{t^3}{3!}e^{x}.
\end{aligned}$$

Then, the approximate solutions of the unknown functions $v(x,t)$ and $w(x,t)$ are given by:

$$\begin{aligned} v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \\ &= v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots \\ &= 1 + e^x - 2t + te^x + 2t + \frac{t^2}{2!}e^x + \frac{t^3}{3!}e^x + \cdots \\ &= 1 + e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \\ &= 1 + e^{x+t}. \end{aligned}$$

$$\begin{aligned} w(x,t) &= \sum_{n=0}^{\infty} w_n(x,t) \\ &= w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + \cdots \\ &= -1 + e^x - 2t - te^x + 2t + \frac{t^2}{2!}e^x - \frac{t^3}{3!}e^x + \cdots \\ &= -1 + e^x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \\ &= e^{x-t} - 1. \end{aligned}$$

Hence, the exact solutions of the unknown functions $v(x,t)$ and $w(x,t)$ are given by:

$$\begin{aligned} v(x,t) &= 1 + e^{x+t} \\ w(x,t) &= e^{x-t} - 1. \end{aligned}$$

The exact solutions are in agreement with the results obtained by LDM in [3].

Example 5.3. Finally, we consider couple system of nonlinear partial differential equation of the form:

$$\begin{aligned} &v_t(x,t) + w(x,t)v_x(x,t) + v(x,t) = 1 \\ &w_t(x,t) - v(x,t)w_x(x,t) - w(x,t) = 1, \end{aligned} \tag{5.27}$$

subject to the initial conditions

$$\begin{aligned} &v(x,0) = e^x \\ &w(x,0) = e^{-x}. \end{aligned} \tag{5.28}$$

Applying the Natural transform on both sides of Eq. (5.27), we obtain:

$$\begin{aligned} &\tfrac{s}{u}v(x,s,u) - \tfrac{1}{u}v(x,0) + \mathbb{N}^+\left[w(x,t)v_x(x,t)\right] + \mathbb{N}^+\left[v(x,t)\right] = \tfrac{1}{s} \\ &\tfrac{s}{u}w(x,t) - \tfrac{1}{u}w(x,t) - \mathbb{N}^+\left[v(x,t)w_x(x,t)\right] - \mathbb{N}^+\left[w(x,t)\right] = \tfrac{1}{s}. \end{aligned} \tag{5.29}$$

Substituting the given initial conditions of Eq. (5.27) into Eq. (5.28), we obtain:

$$\begin{aligned} &v(x,s,u) = \tfrac{e^x}{s} + \tfrac{u}{s^2} - \tfrac{u}{s}\mathbb{N}^+\left[w(x,t)v_x(x,t) + v(x,t)\right] \\ &w(x,s,t) = \tfrac{e^{-x}}{s} + \tfrac{u}{s^2} + \tfrac{u}{s}\mathbb{N}^+\left[v(x,t)w_x(x,t) + w(x,t)\right]. \end{aligned} \tag{5.30}$$

Then by taking the inverse Natural transform of Eq. (5.30), we have:

$$\begin{aligned} v(x,t) &= e^x + t - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[w(x,t)v_x(x,t) + v(x,t)\right]\right] \\ w(x,t) &= e^{-x} + t + \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[v(x,t)w_x(x,t) + w(x,t)\right]\right]. \end{aligned} \tag{5.31}$$

We now assume an infinite series solutions of the unknown functions $v(x,t)$ and $w(x,t)$ of the form:

$$\begin{aligned} v(x,t) &= \textstyle\sum_{n=0}^{\infty} v_n(x,t) \\ w(x,t) &= \textstyle\sum_{n=0}^{\infty} w_n(x,t). \end{aligned} \tag{5.32}$$

Then by using Eq. (5.31), we can re-write Eq. (5.32) in the form:

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} v_n(x,t) &= e^x + t - \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\textstyle\sum_{n=0}^{\infty} A_n(w,v) + \sum_{n=0}^{\infty} v_n(x,t)\right]\right] \\ \textstyle\sum_{n=0}^{\infty} w_n(x,t) &= e^{-x} + t + \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[\textstyle\sum_{n=0}^{\infty} B_n(v,w) + \sum_{n=0}^{\infty} w_n(x,t)\right]\right], \end{aligned} \tag{5.33}$$

where $A_n(w,v)$ and $B_n(v,w)$ are the Adomian polynomial representing the nonlinear terms $w(x,t)v_x(x,t)$ and $v(x,t)w_x(x,t)$ respectively. By comparing both sides of Eq. (5.33), we can easily derive the general recursive relation as follows:

$$\begin{aligned} v_0(x,t) &= e^x + t \\ v_1(x,t) &= -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[A_0(w,v) + v_0(x,t)\right]\right] \\ v_2(x,t) &= -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[A_1(w,v) + v_1(x,t)\right]\right] \\ v_3(x,t) &= -\mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[A_2(w,v) + v_2(x,t)\right]\right]. \end{aligned} \tag{5.34}$$

Then the general recursive relation of the unknown function $v(x,t)$ is given by:

$$v_{n+1}(x,t) = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[A_n(w,v) + v_n(x,t)\right]\right]. \tag{5.35}$$

Similarly, for the unknown function $w(x,t)$, we have:

$$\begin{aligned} w_0(x,t) &= e^{-x} + t \\ w_1(x,t) &= \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[B_0(v,w) + w_0(x,t)\right]\right] \\ w_2(x,t) &= \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[B_1(v,w) + w_1(x,t)\right]\right] \\ w_3(x,t) &= \mathbb{N}^{-1}\left[\tfrac{u}{s}\mathbb{N}^+\left[B_2(v,w) + w_2(x,t)\right]\right]. \end{aligned} \tag{5.36}$$

Thus, the general recursive relation of the unknown function $w(x,t)$ is given by:

$$w_{n+1}(x,t) = \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[B_n(v,w) + w_n(x,t)\right]\right]. \tag{5.37}$$

Therefore, by using the general recursive relations derived in Eq. (5.35) and Eq. (5.37), we can easily the remaining components of the unknown functions $v(x,t)$ and $w(x,t)$ as follows:

$$
\begin{aligned}
v_1(x,t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[A_0(w,v)+v_0(x,t)\right]\right]\\
&= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[w_0(x,t)v_{0x}(x,t)+v_0(x,t)\right]\right]\\
&= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[1+t+e^{x}+te^{x}\right]\right]\\
&= -\left(1+e^{x}\right)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[1\right]\right]-\left(1+e^{x}\right)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t\right]\right]\\
&= -\left(1+e^{x}\right)\mathbb{N}^{-1}\left[\frac{u}{s^2}\right]-\left(1+e^{x}\right)\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right]\\
&= -t-te^{x}-\frac{t^2}{2!}e^{x}-\frac{t^2}{2!}.
\end{aligned}
$$

$$
\begin{aligned}
w_1(x,t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[B_0(v,w)+w_0(x,t)\right]\right]\\
&= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_0(x,t)w_{0x}(x,t)+w_0(x,t)\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[-1+t+e^{-x}-te^{-x}\right]\right]\\
&= \left(e^{-x}-1\right)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[1\right]\right]-\left(e^{-x}-1\right)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t\right]\right]\\
&= \left(e^{-x}-1\right)\mathbb{N}^{-1}\left[\frac{u}{s^2}\right]-\left(e^{-x}-1\right)\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right]\\
&= -t+te^{-x}-\frac{t^2}{2!}e^{-x}+\frac{t^2}{2!}.
\end{aligned}
$$

$$
\begin{aligned}
v_2(x,t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[A_1(w,v)+v_1(x,t)\right]\right]\\
&= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[w_1(x,t)v_{0x}(x,t)+w_0(x,t)v_{1x}(x,t)+v_1(x,t)\right]\right]\\
&= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[-2te^{x}-t+\cdots\right]\right]\\
&= \left(1+2e^{x}\right)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t\right]\right]+\cdots\\
&= \left(1+2e^{x}\right)\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right]+\cdots\\
&= t^2e^{x}+\frac{t^2}{2!}+\cdots.
\end{aligned}
$$

$$\begin{aligned}
w_2(x,t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[B_1(v,w)+w_1(x,t)\right]\right] \\
&= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_1(x,t)w_{0x}(x,t)+v_0(x,t)w_{1x}(x,t)+w_1(x,t)\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[2te^{x}+t+\cdots\right]\right] \\
&= (1+2e^{x})\,\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t\right]\right]+\cdots \\
&= (1+2e^{x})\,\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right]+\cdots \\
&= t^2e^{x}+\frac{t^2}{2!}+\cdots.
\end{aligned}$$

Hence, the approximate solutions of the unknown functions $v(x,t)$ and $w(x,t)$ are given by:

$$\begin{aligned}
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \\
&= v_0(x,t)+v_1(x,t)+v_2(x,t)+\cdots \\
&= e^{x}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right) \\
&= e^{x-t}.
\end{aligned}$$

$$\begin{aligned}
w(x,t) &= \sum_{n=0}^{\infty} w_n(x,t) \\
&= w_0(x,t)+w_1(x,t)+w_2(x,t)+\cdots \\
&= e^{-x}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right) \\
&= e^{t-x}.
\end{aligned}$$

Hence, the exact solutions of the unknown functions $v(x,t)$ and $w(x,t)$ are given by:

$$\begin{aligned}
v(x,t) &= e^{x-t} \\
w(x,t) &= e^{t-x}.
\end{aligned}$$

The result obtained are in excellent agreement with the results obtained by LDM in [3] and by the MLDM in [5].

6. Conclusion

The aim of this work is to introduce a new computational algorithms called the Natural Decomposition Method (NDM) to solve coupled systems of linear and nonlinear partial differential equation. The new method, was successfully used to find exact solutions to all three examples and we compared the results we obtained with the one found by the ADM [1], LDM [3] and MLDM [5]. This clearly shows that the Natural Decomposition Method (NDM) introduces a significant improvement in this fields and can easily be used to solve a wide class of coupled system of linear and nonlinear partial differential equations. Our goal in the future is to apply the NDM to other nonlinear PDEs that arises in other areas of applied science and engineering.

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