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On a New Subclass of Harmonic Univalent Functions Defined by Modified Catas Operator

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Abstract In this paper, we introduce a new class of harmonic univalent functions defined by modified Cata's operator. Coefficient estimates, extreme points, distortion bounds and convex combination for functions belonging to this class are obtained.

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1. Introduction

A continuous complex-valued function $f = u + iv$ is defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [1]).

Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, \qquad |b_1| < 1. \tag{1.2}$$

In [1] Clunie and Shell-Small investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

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Let $\overline{S_H}$ denote the subclasses of S_H consisting of functions $f = h + \overline{g}$ such that h and g given by

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = (-1)^m \sum_{k=1}^{\infty} b_k z^k, \qquad |b_1| < 1. \tag{1.3}$$

For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}$, $\mu \geq 0$ and $l \geq 0$, the extended multiplier transformation $I^m(\mu, l)$ is defined by the following infinite series (see [2]):

$$I^m(\mu, l) f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1 + l + \mu(k-1)}{1 + l}\right]^m a_k z^k. \tag{1.4}$$

Now we can define the modified Catas operator as follows:

$$I(m, \mu, l) f(z) = I^m(\mu, l) h(z) + (-1)^m \overline{I^m(\mu, l) g(z)}, \tag{1.5}$$

where

$$I^m(\mu, l) h(z) = z + \sum_{k=2}^{\infty} \left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m a_k z^k$$

and

$$I^m(\mu, l) g(z) = (-1)^m \sum_{k=1}^{\infty} \left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m b_k z^k.$$

For $0 \leq \alpha < 1$, $\beta \geq 0$, $m, n \in \mathbb{N}_0$, $m > n$, $\delta \in \mathbb{R}$, $\mu, l \geq 0$ and for all $z \in U$, let $SI^{m,n}(\mu, l, \beta, t; \alpha)$ denote the family of harmonic functions $f(z) = h + \overline{g}$, where h and g given by (1.2) and satisfying the analytic criterion

$$\operatorname{Re}\left\{\left(1 + \beta e^{i\delta}\right) \frac{I(m, \mu, l) f(z)}{I(n, \mu, l) f_t(z)} - \beta e^{i\delta}\right\} \geq \alpha, \tag{1.6}$$

where $f_t(z) = (1-t) z + \left(h(z) + \overline{g(z)}\right)$ $(0 \leq t \leq 1)$. Let $\overline{SI}^{m,n}(\mu, l, \beta, t; \alpha)$ be the subclass of $SI^{m,n}(\mu, l, \beta, t; \alpha)$ consisting of functions $f_m = h + \overline{g_m}$ such that h and g given by (1.3).

We note that for suitable choices of β, t, n, m, μ and l, we obtain the following subclasses:

(1) $\overline{SI}^{1,0}(1, 0, 1, 1; \alpha) = G_H(\alpha)$ $(0 \leq \alpha < 1, \delta \in \mathbb{R})$ (see Rosy et al. [3]);

(2) $\overline{SI}^{k+1,k}(1, 0, 1, 1; \alpha) = RH(k, \alpha)$ $(k \in \mathbb{N}, 0 < \alpha \leq 1, \delta \in \mathbb{R})$ (see Yasar and Yalcin [4]);

(3) $\overline{SI}^{m,n}(1, 0, 1, 1; \alpha) = G_{\overline{H}}(m, n, \alpha)$ $(m, n \in \mathbb{N}_0, 0 \leq \alpha < 1, \delta \in \mathbb{R})$ (see Subramanian et al. [5]);

(4) $\overline{SI}^{k+q,k}(1, 0, \beta, 1; \alpha) = \overline{R}_H(k, \alpha, \beta, q)$ $(k \in \mathbb{N}_0, q \in \mathbb{N}, 0 \leq \alpha < 1, \delta \in \mathbb{R})$ (see Dixit et al. [6]);

(5) $\overline{SI}^{m,n}(1, 0, \beta, 1; \alpha) = V_{\overline{H}}(m; n; \alpha; \beta)$ $(m \in \mathbb{N}, n \in \mathbb{N}_0, 0 \leq \beta \leq 1, 0 \leq \alpha < 1, \delta \in \mathbb{R})$ (see Aghalary [7]);

(6) $\overline{SI}^{m,n}(1,0,0,1;\alpha) = \overline{S_H}(m,n,\alpha)$ $(m \in \mathbb{N}, n \in \mathbb{N}_0, 0 \leq \alpha < 1)$
(see Porwal et al. [8]);

(7) $\overline{SI}^{n+1,n}(\mu,0,0,1;\alpha) = \overline{SH}(\mu,n,\alpha)$ $(n \in \mathbb{N}_0, \mu \geq 0, 0 \leq \alpha < 1)$
(see Yasar and Yalcin [4]);

(8) $\overline{SI}^{m,n}(1,0,1,1;\alpha) = TS_H^*(m,n,0,\alpha) = TS_H^*(m,n,\alpha)$ $(m \in \mathbb{N}, n \in \mathbb{N}_0,\ 0 \leq \alpha < 1, \delta \in \mathbb{R})$
(see Sudharsan et al. [9], with $\lambda = 0$);

(9) $\overline{SI}^{m,n}(1,0,0,1;\alpha) = \overline{S}_H(m,n,\alpha,0) = \overline{S}_H(m,n,\alpha)$ $(m \in \mathbb{N}, n \in \mathbb{N}_0,\ 0 \leq \alpha < 1)$
(see Aouf [10], with $\lambda = 0$);

(10) $\overline{SI}^{m,n}(\mu,0,\beta,t;\alpha) = \overline{RS_H}(m,n,\beta,t,\mu,\alpha)$ $(m \in \mathbb{N}, n \in \mathbb{N}_0, \beta \geq 0,\ 0 \leq \mu < 1,\ 0 \leq t \leq 1,$
$0 \leq \alpha < 1)$ (see Porwal et al. [11]);

(11) $\overline{SI}^{1,0}(1,0,\beta,t;\alpha) = \overline{G_H}(\beta,\alpha,t)$ $(\beta \geq 0, 0 \leq t \leq 1, 0 \leq \alpha < 1)$ (see Ahuja et al. [12]).

Also we note that:

(1) Putting $\mu = 1$ and $l = 1$, the class $\overline{SI}^{m,n}(1,1,\beta,t;\alpha)$ reduces to the class reduces to the class

$$\overline{SI}^{m,n}(\beta,t;\alpha) = \left\{ f \in S_H : \operatorname{Re}\left\{ \left(1+\beta e^{i\delta}\right) \frac{I^m f(z)}{I^n f_t(z)} - \beta e^{i\delta} \right\} \geq \alpha, \right.$$

$0 \leq \alpha < 1, \beta \geq 0, 0 \leq t \leq 1, n, m \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}, \delta \in \mathbb{R}, z \in U$,

where I^m is the modified Uralegaddi-Somanatha operator (see [13]), defined as follows:

$$I^m f(z) = I^m h(z) + (-1)^m \overline{I^m g(z)};$$

(2) Putting $\mu = 1$, the class $\overline{SI}^{m,n}(1,l,\beta,t;\alpha)$ reduces to the class

$$\overline{SI}^{m,n}(l,\beta,t;\alpha) = \left\{ f \in S_H : \operatorname{Re}\left\{ \left(1+\beta e^{i\delta}\right) \frac{I_l^m f(z)}{I_l^n f_t(z)} - \beta e^{i\delta} \right\} \geq \alpha, \right.$$

$0 \leq \alpha < 1, \beta \geq 0, 0 \leq t \leq 1, n, m, \delta \in \mathbb{R}, l > -1, z \in U$,

where I_l^m is the modified Cho-Kim operator [14] (also see [15]), defined as follows:

$$I_l^m f(z) = I_l^m h(z) + (-1)^m \overline{I_l^m g(z)}.$$

2. Coefficient Estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $0 \leq \alpha < 1, \beta \geq 0, m \in \mathbb{N}, n \in \mathbb{N}_0, m > n, \delta \in \mathbb{R}, \mu, l \geq 0, 0 \leq t \leq 1$ all $z \in U$.

Theorem 2.1. *Let $f = h + \overline{g}$ be such that $h(z)$ and $g(z)$ given by (1.2). Furthermore, let*

$$\sum_{k=2}^{\infty}\left\{\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t\right\}|a_k|$$
$$+\sum_{k=1}^{\infty}\left\{\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-(-1)^{m-n}\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t\right\}|b_k| \le 1-\alpha. \tag{2.1}$$

Then $f(z)$ is sense-preserving, harmonic univalent in U and $f(z) \in SI^{m,n}(\mu, l, \beta, t; \alpha)$.

Proof. If $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum\limits_{k=1}^{\infty} b_k\left(z_1^k-z_2^k\right)}{\left(z_1^k-zk\right)+\sum\limits_{k=2}^{\infty} a_k\left(z_1^k-z_2^k\right)}\right|$$
$$> 1-\frac{\sum\limits_{k=1}^{\infty} k|b_k|}{1-\sum\limits_{k=2}^{\infty} k|a_k|}$$
$$\ge 1-\frac{\frac{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t}{1-\alpha}|b_k|}{\frac{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t}{1-\alpha}|a_k|}$$
$$\ge 0,$$

which proves univalence. Note that $f(z)$ is sense-preserving in U. This is because

$$\left|h'(z)\right| \ge 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1}$$
$$> 1-\sum_{k=2}^{\infty} k|a_k| \ge \sum_{k=2}^{\infty}\frac{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t}{1-\alpha}|a_n|$$
$$\ge \sum_{k=1}^{\infty}\frac{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t}{1-\alpha}|b_n| \ge \sum_{k=1}^{\infty} k|b_k|$$
$$> \sum_{k=1}^{\infty} k|b_k|\left|z^{k-1}\right| \ge \left|g'(z)\right|.$$

Now we will show that $f(z) \in SI^{m,n}(\mu, l; \beta, t; \alpha)$. We only need to show that if (2.1) holds then the condition (1.6) is satisfied. Using the fact that $Re\{w\} \ge \alpha$ if and only if $|1-\alpha+w| \ge |1+\alpha-w|$, it suffices to show that

$$\left|(1-\alpha)I(n,\mu,l)f_t(z)+\left(1+\beta e^{i\delta}\right)I(m,\mu,l)f(z)-\beta e^{i\delta}I(n,\mu,l)f_t(z)\right|$$
$$-\left|(1+\alpha)I(n,\mu,l)f_t(z)-\left(1+\beta e^{i\delta}\right)I(m,\mu,l)f(z)+\beta e^{i\delta}I(n,\mu,l)f_t(z)\right| \ge 0. \tag{2.2}$$

Substituting for $I^m(\mu,l)\,f(z)$ and $I^n(\mu,l)\,f_t(z)$ in L.H.S. of (2.2) we have

$$= \Bigg|(2-\alpha)\,z + \sum_{k=2}^{\infty}\left\{\left(1-\alpha-\beta e^{i\delta}\right)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t + \left(1+\beta e^{i\delta}\right)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m\right\} a_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty}\left\{\left(1-\alpha-\beta e^{i\delta}\right)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t + (-1)^{m-n}\left(1+\beta e^{i\delta}\right)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m\right\}\overline{b_k z^k}\Bigg|$$

$$- \Bigg|\beta z + \sum_{k=2}^{\infty}\left\{\left(1+\alpha+\beta e^{i\delta}\right)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t - \left(1+\beta e^{i\delta}\right)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m\right\} a_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty}\left\{\left(1+\alpha+\beta e^{i\delta}\right)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t - (-1)^{m-n}\left(1+\beta e^{i\delta}\right)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m\right\}\overline{b_k z^k}\Bigg|$$

$$\geq 2(1-\alpha)|z| - 2\sum_{k=2}^{\infty}\left\{(1+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m - (\alpha+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}|a_k|\,|z|^k$$

$$- 2\sum_{k=1}^{\infty}\left\{(-1)^{m-n}(1+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m - (\alpha+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}|b_k|\,|z|^k$$

$$= \begin{cases} 2(1-\alpha)|z| - 2\sum\limits_{k=2}^{\infty}\left\{(1+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m - (\alpha+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}|a_k|\,|z|^k \\ -2\sum\limits_{k=1}^{\infty}\left\{(1+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m + (\alpha+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}|b_k|\,|z|^k, \text{ if } m-n \text{ is odd} \\ 2(1-\alpha)|z| - 2\sum\limits_{k=2}^{\infty}\left\{(1+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m - (\alpha+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}|a_k|\,|z|^k \\ -2\sum\limits_{k=1}^{\infty}\left\{(1+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^m - (\alpha+\beta)\left[\tfrac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}|b_k|\,|z|^k, \text{ if } m-n \text{ is even} \end{cases}$$

$$= 2(1-\alpha)|z|\left\{1 - \sum_{k=2}^{\infty}\frac{\left\{(1+\beta)\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m-(\alpha+\beta)\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}}{1-\alpha}|a_k|\,|z|^{k-1}\right.$$

$$\left. - \sum_{k=1}^{\infty}\frac{\left\{(1+\beta)\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m-(\alpha+\beta)\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}}{1-\alpha}|b_k|\,|z|^{k-1}\right\}$$

$$> 2(1-\alpha)|z|\left\{1 - \sum_{k=2}^{\infty}\frac{\left\{(1+\beta)\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m-(\gamma+\beta)\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}}{1-\alpha}|a_k|\right.$$

$$\left. - \sum_{k=1}^{\infty}\frac{\left\{(1+\beta)\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m-(\gamma+\beta)\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n t\right\}}{1-\alpha}|b_k|\right\}.$$

The last expression is non negative by (2.1). This completes the proof of Theorem 2.1. The harmonic univalent functions of the form

$$f(z) = z + \sum_{k=2}^{\infty}\frac{1-\alpha}{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t}x_k z^k$$

$$+ \sum_{k=1}^{\infty}\frac{1-\alpha}{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t}\overline{y_k z^k}, \tag{2.3}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (2.1) is sharp. It is worthy to note that the function of the form (2.2) belongs to the class $SI(m,n,\beta,t,\mu,l;\alpha)$ for all $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\leq 1$ because coefficient inequality (2.1) holds. ∎

Theorem 2.2. *A function $f(z)$ of the form (1.3) is in the class $\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$ if and only if*

$$\sum_{k=2}^{\infty}\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)\,t\right\}|a_k|$$
$$+\sum_{k=1}^{\infty}\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)\,t\right\}|b_k|\leq 1-\alpha. \tag{2.4}$$

Proof. Since $\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)\subset SI^{m,n}(\mu,l,\beta,t;\alpha)$, we only need to prove the "only if" part of this theorem. To this end, for functions $f(z)$ of the form (1.3), we notice that the condition

$$\operatorname{Re}\left\{\left(1+\beta e^{i\delta}\right)\frac{I\,(m,\mu,l)\,f(z)}{I\,(n,\mu,l)\,f_t(z)}-\beta e^{i\delta}\right\}\geq\alpha$$

is equivalent to

$$\operatorname{Re}\left\{\frac{\begin{array}{c}(1-\alpha)\,z-\sum_{n=2}^{\infty}\left\{\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}\left(1+\beta e^{i\delta}\right)-\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{n}\left(\alpha+\beta e^{i\delta}\right)t\right\}|a_n|\,z^n\\ +(-1)^{2m-1}\sum_{n=1}^{\infty}\left\{\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}\left(1+\beta e^{i\delta}\right)\right.\\ \left.-(-1)^{m-n}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{n}\left(\alpha+\beta e^{i\delta}\right)t\right\}|b_n|\,\overline{z^n}\end{array}}{z-\sum_{n=2}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{n}t|a_n|z^n+\sum_{n=1}^{\infty}(-1)^{m+n-1}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{n}t|b_n|\overline{z^n}}\right\}$$
$$\geq 0.$$

The above condition must hold for all z, $|z|=r<1$. Choosing the values of z on the positive real axis where $0\leq r<1$, we must have

$$\operatorname{Re}\left\{\frac{\begin{array}{c}(1-\alpha)-\sum_{k=2}^{\infty}\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}\alpha t\right\}|a_k|\,r^{k-1}\\ -\sum_{k=1}^{\infty}\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}\alpha t\right\}|b_k|\,r^{k-1}\end{array}}{z-\sum_{k=2}^{\infty}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}t|a_k|z^k-\sum_{k=1}^{\infty}(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}t|b_k|\overline{z^k}}\right.$$

$$-e^{i\delta}\frac{\sum_{k=2}^{\infty}\beta\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}t\right\}|a_k|\,r^{k-1}-\sum_{k=1}^{\infty}\beta\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}t\right\}|b_k|\,r^{k-1}}{1-\sum_{k=2}^{\infty}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}t|a_k|r^{k-1}-\sum_{k=1}^{\infty}(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}t|b_k|r^{k-1}}\Bigg\}\geq 0.$$

Since $\operatorname{Re}\left(-e^{i\delta}\right)\geq-\left|e^{i\delta}\right|=-1$, the above inequality reduces to

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)\,t\right\}|a_k|\,r^{k-1}-\sum_{k=1}^{\infty}\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)\,t\right\}|b_k|\,r^{k-1}}{1-\sum_{k=2}^{\infty}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}t\,|a_k|\,r^{k-1}-\sum_{k=1}^{\infty}(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}t\,|b_k|\,r^{k-1}}\geq 0. \tag{2.5}$$

If condition (2.3) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence there exist $z_0=r_0$ in $(0,1)$ for which the quotient in (2.5) is negative. This contradicts the required condition for $f(z)\in\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$. This completes the proof of Theorem 2.2. ■

3. Distortion Theorem

Theorem 3.1. *Let the function $f(z)$ defined by (1.3) belong to the class $\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$. Then for $|z|=r<1$, we have*

$$|f(z)|\leq(1+|b_1|)r+\frac{1}{\left[\frac{1+l+\mu}{1+l}\right]^{n}}\left(\frac{(1-\alpha)}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}|b_1|\right)r^2$$

and

$$|f(z)|\geq(1+|b_1|)r-\frac{1}{\left[\frac{1+l+\mu}{1+l}\right]^{n}}\left(\frac{(1-\alpha)}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}|b_1|\right)r^2 \tag{3.1}$$

for $|b_1|\leq\frac{1-\alpha}{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}$. The results are sharp with equality for the functions $f(z)$ defined by

$$f(z)=z+b_1\overline{z}+\frac{1}{\left[\frac{1+l+\mu}{1+l}\right]^{n}}\left(\frac{(1-\alpha)}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}|b_1|\right)\overline{z}^2 \tag{3.2}$$

and

$$f(z)=z-b_1\overline{z}-\frac{1}{\left[\frac{1+l+\mu}{1+l}\right]^n}\left(\frac{(1-\alpha)}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}|b_1|\right)z^2. \tag{3.3}$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f(z)\in\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$. Taking the absolute value of f, we have

$$|f(z)|\le(1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\,r^k\le(1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\,r^2$$

$$=(1+|b_1|)r+\frac{(1-\alpha)}{\left[\frac{1+l+\mu}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu}{1+l}\right]^n(\alpha+\beta)t}\sum_{n=2}^{\infty}\left(\frac{\left[\frac{1+l+\mu}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu}{1+l}\right]^n(\alpha+\beta)t}{(1-\alpha)}|a_k|\right.$$
$$\left.+\frac{\left[\frac{1+l+\mu}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu}{1+l}\right]^n(\alpha+\beta)t}{(1-\alpha)}|b_k|\right)r^2$$

$$\le(1+|b_1|)r+\frac{(1-\alpha)}{\left[\frac{1+l+\mu}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu}{1+l}\right]^n(\alpha+\beta)t}\sum_{k=2}^{\infty}\left(\frac{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t}{(1-\alpha)}|a_k|\right.$$
$$\left.+\frac{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t}{(1-\alpha)}|b_k|\right)r^2$$

$$\le(1+|b_1|)r+\frac{(1-\alpha)}{\left[\frac{1+l+\mu}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu}{1+l}\right]^n(\alpha+\beta)t}\left(1-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{(1-\alpha)}|b_1|\right)r^2$$

$$=(1+|b_1|)r+\frac{1}{\left[\frac{1+l+\mu}{1+l}\right]^n}\left(\frac{(1-\alpha)}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{\left[\frac{1+l+\mu}{1+l}\right]^{m-n}(1+\beta)-(\alpha+\beta)t}|b_1|\right)r^2.$$

The proof of the left hand inequality follows on lines similar to that of the right hand side inequality. This completes the proof of the Theorem 3.1. ∎

Putting $l=0$ in Theorem 3.1, we obtain the following result which modified the result obtained by Porwal et al. [11, Theorem 2.4].

Corollary 3.2. *Let the function $f(z)$ defined by (1.3) belong to the class $\overline{RS_H}(m,n,\beta,t,\mu,\alpha)$. Then for $|z|=r<1$, we have*

$$|f(z)|\le(1+|b_1|)r+\frac{1}{(1+\mu)^n}\left(\frac{(1-\alpha)}{(1+\mu)^{m-n}(1+\beta)-(\alpha+\beta)t}-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{(1+\mu)^{m-n}(1+\beta)-(\alpha+\beta)t}|b_1|\right)r^2$$

and

$$|f(z)|\ge(1+|b_1|)r-\frac{1}{(1+\mu)^n}\left(\frac{(1-\alpha)}{(1+\mu)^{m-n}(1+\beta)-(\alpha+\beta)t}-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{(1+\mu)^{m-n}(1+\beta)-(\alpha+\beta)t}|b_1|\right)r^2$$

for $|b_1| \leq \frac{1-\alpha}{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}$. *The results are sharp with equality for the functions* $f(z)$ *defined by*

$$f(z) = z + b_1\overline{z} + \frac{1}{(1+\mu)^n}\left(\frac{(1-\alpha)}{(1+\mu)^{m-n}(1+\beta)-(\alpha+\beta)t} - \frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{(1+\mu)^{m-n}(1+\beta)-(\alpha+\beta)t}|b_1|\right)\overline{z}^2$$

and

$$f(z) = z - b_1\overline{z} - \frac{1}{(1+\mu)^n}\left(\frac{(1-\alpha)}{(1+\mu)^{m-n}(1+\beta)-(\alpha+\beta)t} - \frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)t}{(1+\mu)^{m-n}(1+\beta)-(\alpha+\beta)t}|b_1|\right)z^2.$$

4. Extreme Points

Theorem 4.1. *Let* $f(z)$ *be given by* (1.3). *Then* $f(z) \in \overline{SI}^{m,n}(\mu, l, \beta, t; \alpha)$ *if and only if*

$$f(z) = \sum_{k=1}^{\infty}(\mu_k h_k(z) + \eta_k g_k(z)), \tag{4.1}$$

where $h_1(z) = z$,

$$h_k(z) = z - \frac{1-\alpha}{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta) - \left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}z^k \tag{4.2}$$

and

$$g_k(z) = z + (-1)^{m-1}\frac{1-\alpha}{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta) - (-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}\overline{z}^k, \tag{4.3}$$

$\mu_k \geq 0, \eta_k \geq 0, \sum\limits_{k=1}^{\infty}(\mu_k + \eta_k) = 1$. *In particular, the extreme points of the class* $\overline{SI}^{m,n}(\mu, l, \beta, t; \alpha)$ *are* $\{h_k\}$ *and* $\{g_k\}$, *respectively.*

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}(\mu_k h_k(z) + \eta_k g_k(z)) \\ &= z + \sum_{k=2}^{\infty}\frac{1-\alpha}{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta) - \left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}\mu_k z^k \\ &\quad + (-1)^m\sum_{k=1}^{\infty}\frac{1-\alpha}{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta) - (-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}\eta_k\overline{z^k}. \end{aligned}$$

Then

$$\sum_{k=2}^{\infty}\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta) - \left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}{1-\alpha} \cdot \left(\frac{1-\alpha}{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta) - \left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}\mu_k\right)$$

$$+\sum_{k=1}^{\infty}\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t\right\}}{1-\alpha}$$

$$\cdot\left(\frac{1-\alpha}{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t\right\}}\eta_k\right)$$

$$=\sum_{k=2}^{\infty}\mu_k+\sum_{k=1}^{\infty}\eta_k=1-\mu_1\leq 1$$

and so $f(z)\in\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$.

Conversely, if $f(z)\in\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$, then

$$|a_k|\leq\frac{1-\alpha}{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t\right\}}$$

and

$$|b_k|\leq\frac{1-\alpha}{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t\right\}}.$$

Setting

$$\mu_k=\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t\right\}}{1-\alpha}|a_k|\ \ (k=2,3,...)$$

and

$$\eta_k=\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{m}(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^{n}(\alpha+\beta)t\right\}}{1-\alpha}|b_k|\ \ (k=1,2,...)\,.$$

Since $0\leq\mu_k\leq 1$ $(k=2,3,...)$ and $0\leq\eta_k\leq 1$ $(k=1,2,...)\,,$ $\mu_1=1-\sum\limits_{k=2}^{\infty}\mu_k-\sum\limits_{k=1}^{\infty}\eta_k\geq 0,$ then, we can see that $f(z)$ can be expressed in the form (4.1). This completes the proof of the Theorem 4.1. ■

5. Convolution and Convex Combination

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}|a_k|\,z^k+\sum_{k=1}^{\infty}|b_k|\,\overline{z^k} \tag{5.1}$$

and

$$F(z)=z+\sum_{k=2}^{\infty}|A_k|\,z^k+\sum_{k=1}^{\infty}|B_k|\,\overline{z^k}, \tag{5.2}$$

the convolution of f and F is given by

$$(f*F)(z)=f(z)*F(z)=z+\sum_{k=2}^{\infty}|a_kA_k|\,z^k+\sum_{k=1}^{\infty}|b_kB_k|\,\overline{z^k}. \tag{5.3}$$

Using this definition, the next theorem shows that the class $\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$ is closed under convolution.

Theorem 5.1. *For $0\le\alpha\le\lambda<1$, let $f\in\overline{SI}^{m,n}(\mu,l,\beta,t;\lambda)$ where $f(z)$ is given by (5.1) and $F\in\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$ where $F(z)$ is given by (5.2). Then $f*F\in\overline{SI}^{m,n}(\mu,l,\beta,t;\lambda)\subset\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$.*

Proof. We wish to show that the coefficients of $f*F$ satisfy the required condition given in Theorem 2.1. For $F\in\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$ we note that $|A_k|\le 1$ and $|B_k|\le 1$. Now, for the convolution function $f*F$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}{1-\alpha}|a_kA_k|\,z^k\\
&+\sum_{k=1}^{\infty}\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}{1-\alpha}|b_kB_k|\,\overline{z^k}\\
\le&\sum_{k=2}^{\infty}\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}{1-\alpha}|a_k|\,z^k\\
&+\sum_{k=1}^{\infty}\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}{1-\alpha}|b_k|\,\overline{z^k}\\
\le&\sum_{k=2}^{\infty}\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}{1-\lambda}|a_k|\,z^k\\
&+\sum_{k=1}^{\infty}\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m(1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n(\alpha+\beta)t\right\}}{1-\lambda}|b_k|\,\overline{z^k}\le 1.
\end{aligned}$$

Therefore $f*F\in\overline{SI}^{m,n}(\mu,l,\beta,t;\lambda)\subset\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$. ∎

Now we show that the class $\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$ is closed under convex combinations of its members.

Theorem 5.2. *The class $\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$ is closed under convex combination.*

Proof. For $i=1,2,3,...$, let $f_i\in\overline{SI}^{m,n}(\mu,l,\beta,t;\alpha)$, where f_i is given by

$$f_i=z+\sum_{k=2}^{\infty}|a_{k_i}|\,z^k+\sum_{k=1}^{\infty}|b_{k_i}|\,\overline{z^k}.$$

Then by using Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m (1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n (\alpha+\beta)t\right\}}{1-\alpha} |a_{k_i}| z^k$$
$$+\sum_{k=1}^{\infty} \frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m (1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n (\alpha+\beta)t\right\}}{1-\alpha} |b_{k_i}| \overline{z}^k \leq 1. \tag{5.4}$$

For $\sum_{k=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{k_i}|\right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{k_i}|\right) \overline{z^k}. \tag{5.5}$$

Then by (5.4), we have

$$\sum_{k=2}^{\infty} \frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m (1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n (\alpha+\beta)t\right\}}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i |a_{k_i}|\right)$$
$$+\frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m (1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n (\alpha+\beta)t\right\}}{1-\alpha} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{k_i}|\right)$$
$$= \sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m (1+\beta)-\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n (\alpha+\beta)t\right\}}{1-\alpha} |a_{k_i}|\right.$$
$$\left.+\sum_{k=1}^{\infty} \frac{\left\{\left[\frac{1+l+\mu(k-1)}{1+l}\right]^m (1+\beta)-(-1)^{m-n}\left[\frac{1+l+\mu(k-1)}{1+l}\right]^n (\alpha+\beta)t\right\}}{1-\alpha} |b_{k_i}|\right)$$
$$\leq \sum_{i=1}^{\infty} t_i = 1.$$

This is the condition required by (2.1) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{SI}^{m,n}(\mu, l, \beta, t; \alpha)$. ■

Remark 5.3. Specializing the parameters β, t, l, μ, n and m, in the above results, we obtain the corresponding results for the corresponding classes $\overline{SI}^{m,n}(\beta, t; \alpha)$ and $\overline{SI}^{m,n}(l, \beta, t; \alpha)$.

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