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On the Darboux Vector of Ruled Surfaces in Galilean Space

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Abstract In this paper, we investigate the Darboux vector of ruled surfaces in Galilean space. There are three types of ruled surfaces in Galilean space. We obtained the relationship between the Darboux and Frenet vectors of each type of ruled surfaces in Galilean space. In addition, an example is constructed and plotted.

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Keywords: ruled surface; Darboux vector; Galilean space

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1. Introduction

The Darboux vector is discovered by French mathematician Gaston Darboux (1842–1917). There are lots of paper which deal with the Darboux vector since, the Darboux vector can be interpreted kinematically as the direction of the instantaneous axis of rotation in the moving trihedron in Euclidean space [1–3]. However, the Darboux vector can be interpreted kinematically as a shear along the absolute line in the pseudo-Galilean space [4]. There are a lot of interesting applications of Darboux vector such as in [5] the authors investigated the robot end-effector motion using the Darboux vector of ruled surface.

The Galilean geometry is well described in [6]. The geometry of ruled surfaces has been largely developed by O. Röschel [7]. He classified ruled surfaces into three types. Some more results about ruled surfaces in $\mathbb{G}_3$ have been given in [8, 9]. A. Öğrenmiş, M. Ergüt and M. Bektaş also described helices in Galilean space [10]. Theory of curves in pseudo-Galilean space described in [11] and the geometry of ruled surface in pseudo-Galilean space $\mathbb{G}_3^1$ has been explained in details in [12].

The Galilean space $\mathbb{G}_3$ is a Cayley–Klein space equipped with the projective metric of signature $(0, 0, +, +)$, as in [9]. The absolute figure of the Galilean geometry consists of

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an ordered triple $\{\omega, f, I\}$, where ω is the real (absolute) plane, f the real line (absolute line) in ω and I the fixed elliptic involution of points of f.

A plane is called Euclidean if it contains f, otherwise it is called isotropic or planes $x = constant$ are Euclidean and so is the plane ω. Other planes are isotropic. A vector $u = (u_1, u_2, u_3)$ is said to non-isotropic if $u_1 \neq 0$. All unit non-isotropic vectors are of the form $u = (1, u_2, u_3)$. For isotropic vectors $u_1 = 0$ holds [9].

The distance between the points $p_i = (x_i, y_i, z_i)$, $i = 1, 2$ is defined by

$$d(p_1, p_2) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2 \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}, & \text{if } x_1 = x_2. \end{cases}$$

Example 1.1. Let $A = (5, 6, 5)$ and $C = (5, 4, 3)$ be two points in Galilean space, shown in Figure 1. The distance between these two points is

$$d_{(A,C)} = \sqrt{(4-6)^2 + (3-5)^2} = 2\sqrt{2}.$$

The Galilean distance is identical with the Euclidean distance.

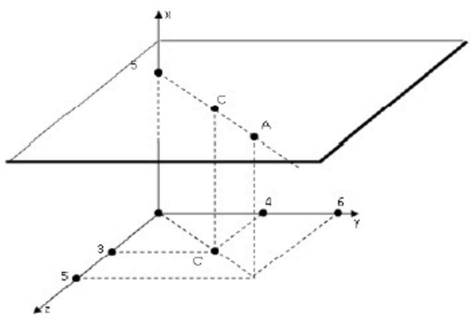

Figure 1.

The Galilean distance between $A = (5, 6, 5)$ and $C' = (0, 4, 3)$ is

$$d_{(A,C')} = |5 - 0| = 5.$$

Also the distance between A and C' in Euclid space is

$$d_{(A,C')} = \sqrt{(0-5)^2 + (4-6)^2 + (3-5)^2} = \sqrt{33}.$$

In $\mathbb{G}^3$ there are four classes of lines:

a) (Proper) non-isotropic lines: they don't meet the absolute line f.
b) (Proper) isotropic lines: lines that don't belong to the plane ω but meet the absolute line f.
c) Unproper non-isotropic lines: all lines of ω but f.
d) The absolute line f.

Definition 1.2. In Figure 2, let ε be plane and $f(\varepsilon)$ the intersection of the absolute line f and ε. The point $f(\varepsilon)$ is called the absolute point of ε. Now, let $f(\varepsilon)^{\perp} = I(f(\varepsilon))$ be the point on f orthogonal to $f(\varepsilon)$ according to the elliptic involution I.

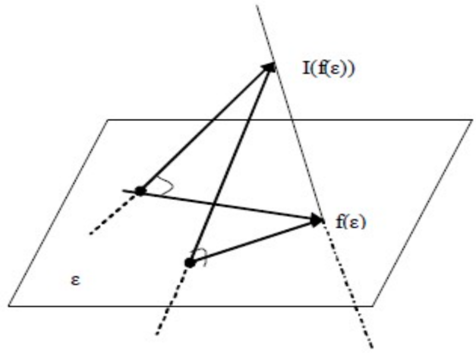

FIGURE 2.

This is elliptic involution because there is no line perpendicular to itself.

Definition 1.3. Let $a = (x, y, z)$ and $b = (x_1, y_1, z_1)$ be vectors in the Galilean space. The scalar product is defined by

$$< a, b >= x_1 x. \tag{1.1}$$

If $< a, p >= 0$ then $a \perp p$ (in the sense of the Galilean geometry) implies, $a^2 \neq 0$ that $p = (0, y, z)$ is an isotropic vector.

The scalar product of two isotropic vectors $p = (0, y, z)$ and $q = (0, y_1, z_1)$ is defined by

$$< p, q >_1= yy_1 + zz_1. \tag{1.2}$$

In Figure 3, let $\overrightarrow{AB} = (0, 2, 2)$ and $\overrightarrow{AC} = (5, 2, 2)$ be two vectors in Galilean space. AB is perpendicular to the AC in the sense of Galilean geometry.

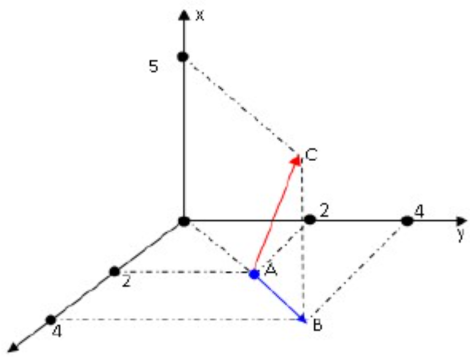

FIGURE 3.

Definition 1.4. Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ be vectors in the Galilean space. The vector product is defined by

$$u \wedge v = \begin{vmatrix} 0 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (0, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \tag{1.3}$$

in Galilean space $\mathbb{G}^3$ [9].

Definition 1.5. An admissible curve is given by the parametrization

$$r(u) = (u, y(u), z(u)). \tag{1.4}$$

In Figure 4, the associated invariant moving trihedron is given by

$$\begin{aligned} \mathbf{t} &= (1, y'(u), z'(u)), \\ \mathbf{n} &= \frac{1}{\kappa}(0, y''(u), z''(u)), \\ \mathbf{b} &= \frac{1}{\kappa}(0, -z''(u), y''(u)) \end{aligned} \tag{1.5}$$

where $\kappa = \sqrt{y''(u)^2 + z''(u)^2}$ is the curvature.

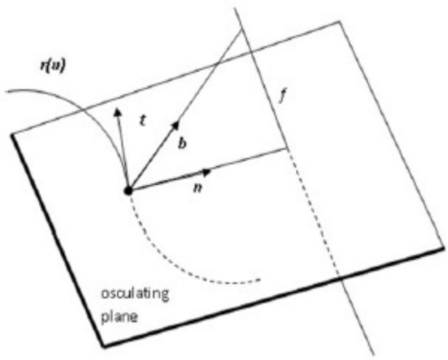

FIGURE 4.

Frenet formulas may be written as

$$\frac{d}{du}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{1.6}$$

where $\tau = \dfrac{1}{\kappa^2}\det[r'(u), r''(u), r'''(u)]$ is the torsion.

According to the absolute figure of $\mathbb{G}^3$, there are three types of ruled surfaces in $\mathbb{G}^3$:

Type A: Non-conoidal or conoidal ruled surfaces whose striction line does not lie in a Euclidean plane.
Type B: Ruled surfaces with the striction line in a Euclidean plane.
Type C: Conoidal ruled surfaces with the absolute line as the directional line in infinity [8].

2. Darboux Vector of Ruled Surface of Type A

Definition 2.1. Let Φ_A be a ruled surface of type A in $\mathbb{G}_3$ given by the parametrization

$$\Phi_A(x, u) = m(x) + ue(x) \tag{2.1}$$

where $m(x) = (x, y(x), z(x))$ is the directrix and $e(x) = (1, a_2(x), a_3(x))$ is the direction unit vector.

The associated trihedron is defined by

$$\begin{aligned} \mathbf{t} &= (1, a_2, a_3), \\ \mathbf{n} &= \frac{1}{\kappa}(0, a_2', a_3'), \\ \mathbf{b} &= \frac{1}{\kappa}(0, -a_3', a_2') \end{aligned} \tag{2.2}$$

where $\kappa = \sqrt{(a_2')^2 + (a_3')^2}$.

Frenet formulas are given as follows

$$\frac{d}{dx}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{2.3}$$

where $\tau = \dfrac{\det(\alpha, \alpha', \alpha'')}{\kappa^2}$ is called the torsion.

The Frenet vector is

$$f_A = \tau \mathbf{t} + \kappa \mathbf{b} \tag{2.4}$$

which satisfies $\dfrac{d\mathbf{t}}{dx} = f_A \wedge \mathbf{t}$, $\dfrac{d\mathbf{n}}{dx} = f_A \wedge \mathbf{n}$ and $\dfrac{d\mathbf{b}}{dx} = f_A \wedge \mathbf{b}$.

The surface frame $[\mathbf{O}, \mathbf{S}_n, \mathbf{S}_b]$ is defined by

$$\mathbf{O} = \mathbf{t}, \quad \mathbf{S}_n = \frac{\Phi_x \wedge \Phi_u}{|\Phi_x \wedge \Phi_u|}, \quad \mathbf{S}_b = \mathbf{O} \wedge \mathbf{S}_n. \tag{2.5}$$

Let φ be the Euclidean angle between the isotropic vectors $\mathbf{S}_n$ and $\mathbf{n}$, we have

$$\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{2.6}$$

From (2.6), we obtain

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix}. \tag{2.7}$$

Differentiating (2.6), then substituting (2.3) and (2.7) into the result gives

$$\frac{d}{dx}\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix} = \begin{bmatrix} 0 & \kappa\cos\varphi & -\kappa\sin\varphi \\ 0 & 0 & d\varphi + \tau \\ 0 & -d\varphi - \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix}. \tag{2.8}$$

Moreover, the geodesic curvature k_g, the normal curvature k_n and the relative torsion τ_g are defined by

$$\begin{aligned} k_n &= \kappa\cos\varphi, \\ k_g &= -\kappa\sin\varphi, \\ \tau_g &= d\varphi + \tau. \end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.8), one gets

$$\frac{d}{dx}\begin{bmatrix}\mathbf{O}\\ \mathbf{S}_n\\ \mathbf{S}_b\end{bmatrix}=\begin{bmatrix}0 & k_n & k_g\\ 0 & 0 & \tau_g\\ 0 & -\tau_g & 0\end{bmatrix}\begin{bmatrix}\mathbf{O}\\ \mathbf{S}_n\\ \mathbf{S}_b\end{bmatrix}. \tag{2.10}$$

The Darboux vector of the ruled surface of type A is

$$U_A=\tau_g\mathbf{O}-k_n\mathbf{S}_n+k_g\mathbf{S}_b \tag{2.11}$$

which satisfies $\dfrac{d\mathbf{O}}{dx}=U_A\wedge\mathbf{O}$, $\dfrac{d\mathbf{S}_n}{dx}=U_A\wedge\mathbf{S}_n$ and $\dfrac{d\mathbf{S}_b}{dx}=U_A\wedge\mathbf{S}_b$.

It is easy to see that

$$\mathbf{t}'=U_A\wedge\mathbf{O}=f_A\wedge\mathbf{t}, \tag{2.12}$$

which implies that

$$U_A=f_A+\lambda\mathbf{t}. \tag{2.13}$$

From (2.4), (2.11) and (2.13), one gets

$$\tau_g\mathbf{O}-k_n\mathbf{S}_n+k_g\mathbf{S}_b=\tau\mathbf{t}+\kappa\mathbf{b}+\lambda\mathbf{t}. \tag{2.14}$$

Since $\mathbf{O}$ is coincident with the $\mathbf{t}$, we have

$$\tau_g=\tau+\lambda. \tag{2.15}$$

Using (2.9), (2.13) and (2.15), one obtains the relationship between the Darboux and Frenet vectors of the ruled surface of type A in the following form

$$U_A=f_A+d\varphi\mathbf{t}. \tag{2.16}$$

3. Darboux Vector of Ruled Surface of Type B

Definition 3.1. A ruled surface of type B can be parametrized by

$$\Phi_B(x,u)=r(x)+ue(x) \tag{3.1}$$

where its striction curve $r(x)=(0,y(x),z(x))$ lies in a Euclidean plane in Galilean space. $e(x)=(1,a_2(x),a_3(x))$ is the direction unit vector.

The associated trihedron of the ruled surfaces of type B is defined by

$$\begin{aligned}\mathbf{t}&=(1,a_2(x),a_3(x)),\\ \mathbf{n}&=(0,-z'(x),y'(x)),\\ \mathbf{b}&=(0,y'(x),z'(x)).\end{aligned} \tag{3.2}$$

Then the Frenet formulas are

$$\frac{d}{dx}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ 0 & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}. \tag{3.3}$$

The Frenet vector is obtained by

$$f_B=-\tau\mathbf{t}-\kappa\mathbf{b}. \tag{3.4}$$

The surface frame $[\mathbf{O},\mathbf{S}_n,\mathbf{S}_b]$ is defined by

$$\mathbf{O}=\mathbf{t},\quad \mathbf{S}_n=\frac{\Phi_x\wedge\Phi_u}{|\Phi_x\wedge\Phi_u|},\quad \mathbf{S}_b=\mathbf{O}\wedge\mathbf{S}_n. \tag{3.5}$$

Let φ be the Euclidean angle between the isotropic vectors $\mathbf{S}_n$ and $\mathbf{n}$, we have

$$\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & \sin\varphi & -\cos\varphi \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{3.6}$$

From (3.6), we have

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & \sin\varphi & -\cos\varphi \end{bmatrix} \begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix}. \tag{3.7}$$

Differentiating (3.6), then substituting (3.3) and (3.7) into the result, we obtain

$$\frac{d}{dx}\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix} = \begin{bmatrix} 0 & \kappa\cos\varphi & \kappa\sin\varphi \\ 0 & 0 & -d\varphi-\tau \\ 0 & d\varphi+\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix}. \tag{3.8}$$

Moreover, the geodesic curvature k_g, the normal curvature k_n and the relative torsion τ_g are defined by

$$\begin{aligned} k_n &= \kappa\cos\varphi, \\ k_g &= \kappa\sin\varphi, \\ \tau_g &= d\varphi+\tau. \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8), finally one gets

$$\frac{d}{dx}\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix} = \begin{bmatrix} 0 & k_n & k_g \\ 0 & 0 & -\tau_g \\ 0 & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix}. \tag{3.10}$$

The Darboux vector of the ruled surface of type B is obtained by

$$U_B = -\tau_g\mathbf{O} - k_n\mathbf{S}_n + k_g\mathbf{S}_b. \tag{3.11}$$

One obtains the relationship between the Darboux and Frenet vectors of the ruled surface of type B in the following form

$$U_B = f_B - d\varphi\mathbf{t}. \tag{3.12}$$

4. Darboux Vector of Ruled Surface of type C

Definition 4.1. A ruled surface of type C can be parametrized by

$$\Phi_C(x,u) = r(x) + ua(x) \tag{4.1}$$

where $r(x) = (x, y(x), 0)$ is called the directrix and $a(x) = (0, a_2(x), a_3(x))$ is the direction unit vector.

The associated orthonormal trihedron is given by

$$\begin{aligned} \mathbf{t} &= (1, y'(x), 0), \\ \mathbf{n} &= (0, a_2(x), a_3(x)), \\ \mathbf{b} &= (0, -a_3(x), a_2(x)). \end{aligned} \tag{4.2}$$

Let θ be the Euclidean angle between $z = 0$ plane and $\mathbf{n}$, then the Frenet formulas are obtained by

$$\frac{d}{dx}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa\cos\theta & -\kappa\sin\theta \\ 0 & 0 & \frac{1}{\delta} \\ 0 & -\frac{1}{\delta} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{4.3}$$

where $\kappa = y''$ and $\delta = -\frac{a_3}{a_2'}$ are called as curvature and torsion, respectively.

The Frenet vector of the associated trihedron of ruled surface of type C is

$$f_C = \frac{1}{\delta}\mathbf{t} + \kappa\sin\theta\mathbf{n} + \kappa\cos\theta\mathbf{b}. \tag{4.4}$$

The surface frame $[\mathbf{O}, \mathbf{S}_n, \mathbf{S}_b]$ is defined as

$$\mathbf{O} = \mathbf{t}, \quad \mathbf{S}_n = \frac{\Phi_x \wedge \Phi_u}{|\Phi_x \wedge \Phi_u|}, \quad \mathbf{S}_b = \mathbf{O} \wedge \mathbf{S}_n. \tag{4.5}$$

Let φ be the Euclidean angle between the isotropic vectors $\mathbf{S}_n$ and $\mathbf{n}$, we have

$$\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{4.6}$$

From (4.6), we have

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix}. \tag{4.7}$$

Differentiating (4.6), then substituting (4.3) and (4.7) into the result, finally we get

$$\frac{d}{dx}\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix} = \begin{bmatrix} 0 & \kappa\cos\phi & -\kappa\sin\phi \\ 0 & 0 & d\varphi + \frac{1}{\delta} \\ 0 & -d\varphi - \frac{1}{\delta} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix} \tag{4.8}$$

where

$$\varphi + \psi = \phi. \tag{4.9}$$

Moreover, the geodesic curvature k_g, the normal curvature k_n and the relative torsion τ_g are defined by

$$\begin{aligned} k_n &= \kappa\cos\phi, \\ k_g &= -\kappa\sin\phi, \\ \tau_g &= d\varphi + \frac{1}{\delta}. \end{aligned} \tag{4.10}$$

Substituting (4.10) into (4.8), we have

$$\frac{d}{dx}\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix} = \begin{bmatrix} 0 & k_n & k_g \\ 0 & 0 & \tau_g \\ 0 & -\tau_g & 0 \end{bmatrix}\begin{bmatrix} \mathbf{O} \\ \mathbf{S}_n \\ \mathbf{S}_b \end{bmatrix}. \tag{4.11}$$

The Darboux vector is obtained by

$$U_C = \tau_g \mathbf{O} - k_n \mathbf{S}_n + k_g \mathbf{S}_b. \tag{4.12}$$

Consequently, we get the relationship between the Frenet and Darboux vectors of ruled surface of type C in the following form

$$U_C = f_C + \frac{1}{\delta}\mathbf{t}. \tag{4.13}$$

Example 4.2. In Figure 5, let the ruled surface of type A be parametrized by

$$\Phi_A(x,u) = \left(\frac{p}{B}x, A\cos\frac{x}{B}, -A\sin\frac{x}{B}\right) + u\left(1, B\sin\frac{x}{B}, B\cos\frac{x}{B}\right) \tag{4.14}$$

where $A \neq 0, B \neq 0, p \neq 0$.

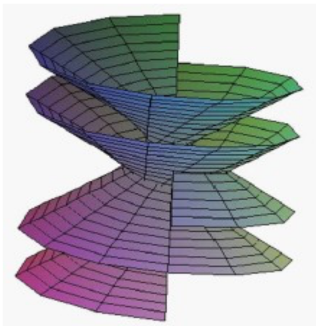

FIGURE 5.

The associated trihedron is defined by

$$\begin{aligned} \mathbf{t} &= \left(1, B\sin\frac{x}{B}, B\cos\frac{x}{B}\right), \\ \mathbf{n} &= \left(0, \cos\frac{x}{B}, -\sin\frac{x}{B}\right), \\ \mathbf{b} &= \left(0, \sin\frac{x}{B}, \cos\frac{x}{B}\right). \end{aligned} \tag{4.15}$$

Then the Frenet formulas are obtained as follows

$$\frac{d}{dx}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{B} \\ 0 & \frac{1}{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{4.16}$$

where

$$\kappa = 1, \quad \tau = -\frac{1}{B}. \tag{4.17}$$

The Frenet vector is

$$f_A = \left(-\frac{1}{B}, 0, 0\right). \tag{4.18}$$

Using (2.9) and (4.17) gives

$$\begin{aligned} k_g &= \cos\varphi, \\ k_n &= -\sin\varphi, \\ \tau_g &= d\varphi - \frac{1}{B}. \end{aligned} \tag{4.19}$$

Substituting (4.19) into (2.11), we have

$$U_A = \left(d\varphi - \frac{1}{B}\right)\mathbf{O} + \sin\varphi\mathbf{S}_n + \cos\varphi\mathbf{S}_b.$$

Using (2.7) gives

$$U_A = \left(d\varphi - \frac{1}{B}\right)\mathbf{t} + \mathbf{b},$$

which implies that

$$U_A = f_A + d\varphi\mathbf{t}.$$

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References

[1] H.H. Uğurlu, H. Kocayiğit, The Frenet and Darboux instantaneous rotation vectors of curves on timelike surface, Math. Comput. Appl. 1 (2) (1996) 133–141.

[2] A. Yücesan, A.C. Çöken, N. Ayyıldız, On the Darboux rotation axis of Lorentz space curve, Appl. Math. Comp. 155 (2) (2004) 345–351.

[3] W. Barthel, Zum Drehvorgang Der Darboux-Achse Einer Raumkurve, Journal of Geometry 49 (1994) 46–49.

[4] C. Ekici, M. Dede, On the Darboux vector of ruled surfaces in pseudo Galilean space, Math. and Comp. App. 16 (2011) 830–838.

[5] B.S. Ryuh, G.R. Pennock, Accurate motion of a robot end-effector using the curvature theory of ruled surfaces, Journal of Mechanisms, Transmissions and Automation in Design 110 (4) (1988) 383–388.

[6] I.M. Yaglom, A Simple Non-Euclidean Geometry and Its Physical Basis, Springer-Verlag New York Inc., 1979.

[7] O. Röschel, Die Geometrie des Galileischen Raumes, Habilitationsschrift, Leoben, 1984.

[8] I. Kamenarovic, Existence theorems for ruled surfaces in the Galilean space $\mathbb{G}_3$, Rad Hrvatske akad. znan. umj. mat. 456 (1991) 183–196.

[9] Z. Milin-Sipus, Ruled Weingarten surfaces in Galilean space, Periodica Mathematica Hungarica 56 (2) (2008) 213–225.

[10] A. Öğrenmiş, M. Ergüt, M. Bektaş, On the Helices in the Galilean Space $\mathbb{G}_3$, Iranian Journal of Science & Technology, Transaction A, Vol. 31, No. A2, printed in: The Islamic Republic of Iran, 2007.

[11] B. Divjak, Curves in pseudo-Galilean geometry, Annales Univ. Sci. Budapest 41 (1998) 117–128.

[12] B. Divjak, Z. Milin-Sipus, Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces, Acta Math. Hungar. 98 (3) (2003) 203–215.