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The planar soap bubble problem with equal pressure regions

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Abstract : The planar soap bubble problem seeks the least-perimeter way to enclose and separate regions of m given areas in $\mathbb{R}^2$. We study the possible configurations for perimeter minimizing enclosures for more than three regions. For four and five regions, we prove that a perimeter minimizing enclosure with equal pressure regions must have connected.

1 Introduction

A lot of peoples believe in that the soap-bubble can enclose and separate the air by use the least surface. The ancient Greeks believed in that a circle is the best way to enclose a single given area but they could not proved it until much later in the late nineteenth century. In 1993, Foisy, Alfaro, Brock, Hodges and Zimba [4] solved the plannar double bubble problem. A bubble is *standard* if every region is connected. For the case of three areas, Vaughn [6] proved in Ph.D. thesis that any minimizing triple bubble with eqaul pressures and without empty chambers is standard. The plannar triple bubble conjecture was complete proved by Wichilamala ([7], [8]). For m-bubble, $m > 3$, problems until open.

2 Preliminaries

An m-bubble can be consider as an embedded graph on the plane where each face is labeled by a number $1, \dots, m$ or 0.

Theorem 2.1 ([1], [5], [3]). *For $A_1, \dots, A_m > 0$, there is a minimizing cluster of areas $A_1, \dots, A_m$. Every minimizing cluster* (1) *is composed of finitely many circular/straight edges separating different regions and meeting only in threes at* $120°$ *angles.* (2) *All edges form a connected graph.* (3) *There are pressures $p_1, \dots, p_m \in \mathbb{R}$ such that every edge between R_i and R_j has curvature $p_i - p_j$* (*curves into the lower pressure region*) *where p_0 is set to be zero.*

Proposition 2.2 ([3]). *For a bubble B with pressures $p_1, \dots, p_m$, and any variation $\{B_t\}$ of B, we have*

$$\left.\frac{dlength\,(B_t)}{dt}\right|_0 = \sum_{i=1}^{m} p_i \left.\frac{dA_i\,(t)}{dt}\right|_0$$

where $A_i\,(t)$ *denotes the area of the* i^{th} *bounded region of* B_t.

A bubble is called *stationary* if it has no area-preserving variation that initially decreases length. A bubble is called *stable* if it is stationary and has no area-preserving variation that decreases length in second order.

Proposition 2.3 ([3]). *A stationary bubble of areas* $A_1, \ldots, A_m$ *and pressures* $p_1, \ldots, p_m$ *has total length* $2\sum p_i A_i$.

We can easily see that if all pressure are equal then the pressure is positive.

Proposition 2.4 ([2]). *For a minimizing bubble, any two components may meet at most once, along a single edge.*

Corollary 2.5 ([4]). *A minimizing* m*-bubble has no* 2*-sided component if* $m \geqq 3$.

From the weak approach ([7], [8]), *weak minimizers* refer to perimeter minimizing bubbles without empty chambers.

Theorem 2.6 ([7], [8]). *For* $m \leqq 6$*, the planar* m*-bubble conjecture holds if every weak minimizer is standard.*

Theorem 2.7 ([7], [8]). *A stable* m*-bubble has at most* m *disjoint nonhexagonal convex components.*

Lemma 2.8 ([7], [8]). *For an* n*-sided component of a bubble, the sum of all edges' turning angles is* $\frac{6-n}{3}\pi$ *if the component is bounded and* $\frac{-6-n}{3}\pi$ *if the component is unbounded.*

Lemma 2.9 ([6]). *In a weakly minimizing* m*-bubble with equal pressures, any* n*-sided component have at most* 6 *sides. In addition, any* n*-sided component that share an edge with the exterior region have at most* 5 *sides.*

Lemma 2.10 ([6]). *In a minimizing* m*-bubble with equal pressures, there is a unique shape for a* 3*-sided component, a one parameter family of possible* 4*-sided component, and a two parameter family of possible* 5*-sided component.*(*See Figure* 1*,* 2 *and* 3)

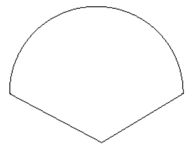

Figure 1: The unique shape for a 3-sided component (Figure 4.1 in [6])

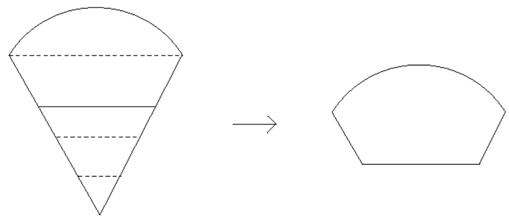

Figure 2: A choice determine a 4-sided component (Figure 4.2 in [6])

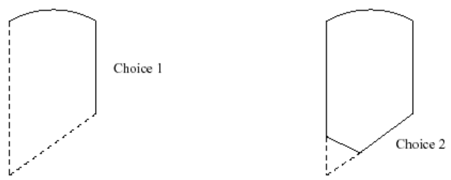

Figure 3: Two choices determine a 5-sided component (Figure 4.3 in [6])

Corollary 2.11. *If both adjacent components of a 3-sided component in a minimizing m-bubble with equal pressures are 5-sided, then they are equal, so we can exchange labels of them.*

Proof. This follows from Lemma 2.10. □

Corollary 2.12. *Every consecutive 4-sided components in a minimizing m-bubble with equal pressures are equal, so we can exchange labels of them.*

Proof. This follows from Lemma 2.10. □

Corollary 2.13. *Let B be a weakly minimizing m-bubble with equal pressures and E an exterior edge in B. Then E intersects the boundary of the convexhull of B.*

Proof. This follows from Lemma 2.9 and 2.10. □

3 Bubbles with equal pressures

Now, we set the radius is 1 and $m > 3$ and denote the number of n-sided components by N_n.

Lemma 3.1. *A region of a weakly minimizing m-bubble with equal pressures has at most one 3-sided component.*

Proof. By the fixed length in Figure 4, the area of a 3-sided component less than $2d$. Thus, we can decrease the length and balance areas by nibbles. By Corollary 2.13, the new edge cannot meet the other part of the bubble.

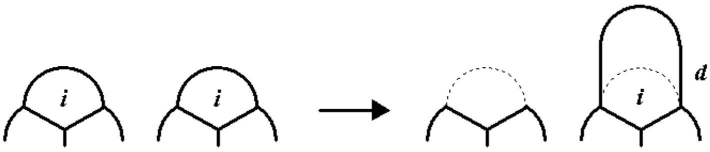

Figure 4: Some regular bubbles improve to nonregular bubbles

□

Lemma 3.2. *A region of a weakly minimizing m-bubble with equal pressures can not has both a 3-sided component and a 4-sided component .*

Proof. By the fixed length in Figure 5, the area of a 3-sided component less than $\sqrt{3}d$. Thus, using Corollary 2.13, we can decrease the length and balance areas by nibbles.

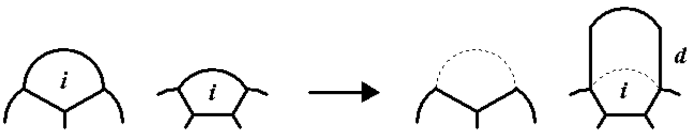

Figure 5: Some regular bubbles improve to nonregular bubbles

□

Lemma 3.3. *A region of a weakly minimizing m-bubble with equal pressures has at most one 4-sided component.*

Proof. By the fixed length in Figure 6, the area of a 4-sided component $< \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) + \frac{\sqrt{3}}{4}\left(\sqrt{3}\right)^2 < \frac{\pi}{3} + \sqrt{3}$. Thus, using Corollary 2.13, we can decrease the length and balance areas by nibbles.

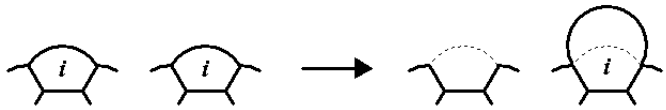

Figure 6: Some regular bubbles improve to nonregular bubbles

□

Theorem 3.4. *A weakly minimizing m-bubble with equal pressures has $N_3+N_4 \leqq m$. Moreover, their label are different.*

Proof. This follows from Lemma 3.1, 3.2 and 3.3. □

Lemma 3.5. *A minimizing m-bubble with equal pressures has $2N_3 + N_4 = 6$.*

Proof. This follows from Lemma 2.8. □

Lemma 3.6. *Let $m \geqq 4$. If a weakly minimizing m-bubble with equal pressures has $N_3 = 2$, $N_4 = 2$ and $N_6 = 0$, then $N_5 = m - 4$. (See Examples in Figure 7)*

Proof. By Theorem 3.4, every 3-sided and 4-sided component has different labels. If $N_5 = 1$, we can not label the 5-sided component for $m = 4$. Suppose that $N_5 > m - 4$ and $N_5 > 1$. If both adjacent components of a 3-sided component are 5-sided, then $N_6 > 0$. Now, each 3-sided component is adjacent with a 4-sided component and a 5-sided component. In this assumption, every 5-side component are equal, so we can exchange labels of them until we found two consecutive components with a label. This is a contradiction. □

Figure 7: Some bubbles with equal pressures which $N_3 = 2$, $N_4 = 2$ and $N_6 = 0$.

Lemma 3.7. *Let $m \in \{4, 5\}$. If an m-bubble with equal pressures has $N_3 = 2$, $N_4 = 2$ and $N_6 = 1$, then it is not weakly minimizing.*

Proof. By Theorem 2.7, all possible combinatorial types are in Figure 8. By Theorem 3.4, every 3-sided and 4-sided component has different labels. Since every 5-side component in Figure 8 are equal, we can exchange labels of them until we found two consecutive components with a label. □

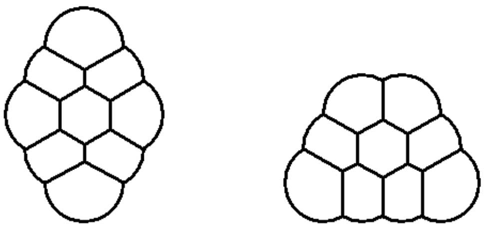

Figure 8: Some bubbles which has one hexagonal component.

Lemma 3.8. *Let $m \in \{4,5\}$. If a m-bubble with equal pressures has $N_3 = 3$ and $N_4 = 0$, then it is not weakly minimizing.*

Proof. By Theorem 2.7, all possible combinatorial types are in Figure 9. By Lemma 3.1, every 3-sided component has different labels. Since every 5-side component in Figure 9 are equal, we can exchange labels of them until we found two consecutive components with a label. □

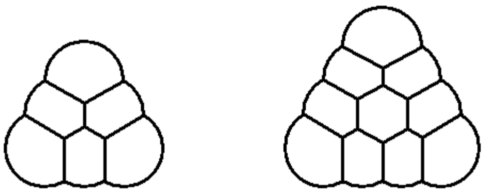

Figure 9: Some bubbles which has three 3-sided component.

Lemma 3.9. *A region of a weakly minimizing m-bubble with equal pressures can not has both a 3-sided component and a 5-sided component .*

Proof. By the fixed length in Figure 10, the area of a 3-sided component less than $2d + \left[\tan\frac{\pi}{12} - 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right]$. Thus, using Corollary 2.13, we can decrease the length and balance areas by nibbles.

□

Lemma 3.10. *A region of a weakly minimizing m-bubble with equal pressures can not has both a 4-sided component that the length of the bottom more than $\frac{2}{\sqrt[4]{3}}\sqrt{\frac{\pi}{2} - \tan\frac{\pi}{12}}$ and a 5-sided component .*

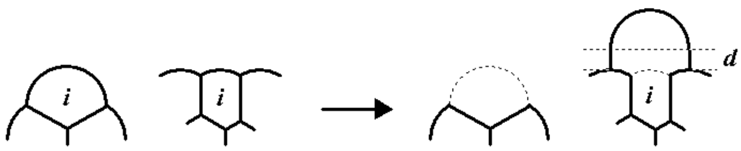

Figure 10: Some regular bubbles improve to nonregular bubbles

Proof. By the fixed length in Figure 11, the area of a 4-side component $< \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) + \frac{\sqrt{3}}{4}\left[3 - \left(\frac{2}{\sqrt[4]{3}}\sqrt{\frac{\pi}{2} - \tan\frac{\pi}{12}}\right)^2\right] < \frac{\pi}{3} + \sqrt{3}$. Thus, using Corollary 2.13, we can decrease the length and balance areas by nibbles.

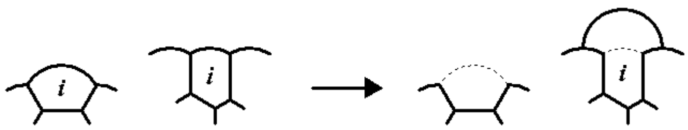

Figure 11: Some regular bubbles improve to nonregular bubbles

□

Lemma 3.11. *A weakly minimizing m-bubble with equal pressures which has a 4-sided component of R_i with a 5-sided component of R_j adjacent on one side, it must has not a 4-sided component of R_j with a 5-sided component of R_i adjacent on one side.*

Proof. By Lemma 3.10, we can decrease the length and balance areas as see in Figure 12.

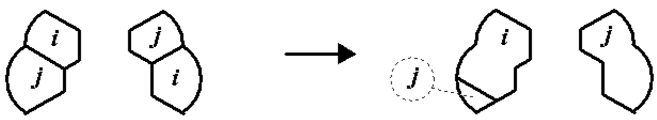

Figure 12: Some regular bubbles improve to nonregular bubbles

□

Proposition 3.12. *Every weakly minimizing 4-bubble with equal pressures is standard.*

Proof. By Theorem 3.4, we have $N_3 + N_4 \leqq 4$. By Lemma 3.5 and 3.8, we have $N_3 = 2$ and $N_4 = 2$. We will divide into cases according to N_6.

Case $N_6 = 0$. By Lemma 3.6, we have $N_5 = 0$. The possibility is in Figure 13.

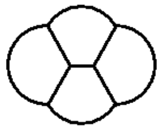

Figure 13: The standard 4-bubble.

Case $N_6 = 1$. By Lemma 3.7, all possibilities are not weakly minimizing.

Case $N_6 > 1$. By Theorem 2.7, every weak minimizer has at most 4 disjoint nonhexagonal convex components. The possibility is in Figure 14. By Theorem 3.4 and Lemma 3.9, we label it in Figure 15 which is a contradiction with Lemma 3.11.

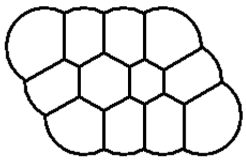

Figure 14: A 4-bubble which has two hexagonal components.

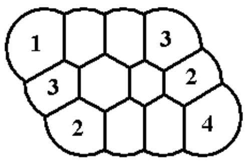

Figure 15: A 4-bubble which has two hexagonal components.

□

Proposition 3.13. *Every weakly minimizing* 5*-bubble with equal pressures is standard.*

Proof. By Theorem 3.4, we have $N_3 + N_4 \leqq 5$. By Lemma 3.5 and 3.8, we have two cases.

*Case*1 : $N_3 = 2$ and $N_4 = 2$. We will divide into subcases according to N_6.

$Subcase 1A : N_6 = 0$. By Lemma 3.6, we have $N_5 = 1$. The possibility is in Figure 16.

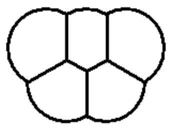

Figure 16: The standard 5-bubble.

$Subcase 1B : N_6 = 1$. By Lemma 3.7, all possibilities are not weakly minimizing.

$Subcase 1C : N_6 > 1$. By Theorem 2.7, every weak minimizer has at most 5 disjoint nonhexagonal convex components. All possible combinatorial types are in Figure 17. Now, every 3-sided and4-sided component has different labels. Consider possibility (a), (b) and (c). By Corollary 2.11 and Lemma 3.9, we label them in Figure 18 (a), (b) and (c) which is a contradiction with Lemma 3.11. Consider possibility (d). By Corollary 2.11, 2.12 and Lemma 3.9, we label it in Figure 18 (d). Thus, $\{\alpha, \beta, \gamma\} = \{1, 2\}$ which is a contradiction.

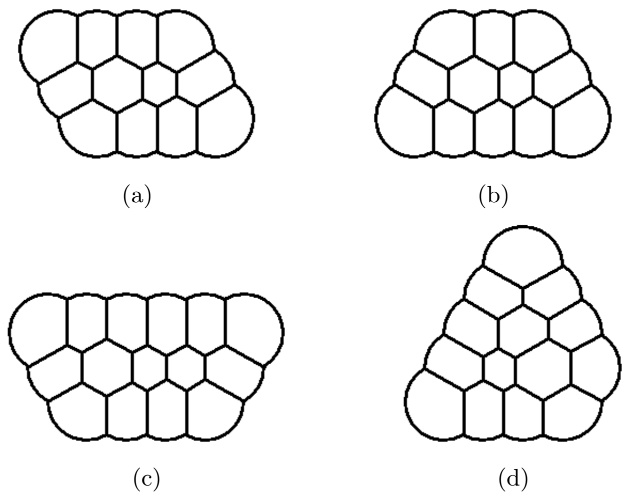

Figure 17: Some 5-bubble which has two or three hexagonal components.

$Case 2 : N_3 = 1$ and $N_4 = 4$. We will divide into subcases according to N_6.

$Subcase 2A : N_6 \leqq 2$. All possible combinatorial types are in Figure 19. Now, every 3-sided and4-sided component has different labels. We label them in Figure 20. By Corollary 2.12 and Lemma 3.9, we can not label i.

$Subcase 2B : N_6 > 2$. By Theorem 2.7, every weak minimizer has at most 5 disjoint nonhexagonal convex components. All possible combinatorial types are

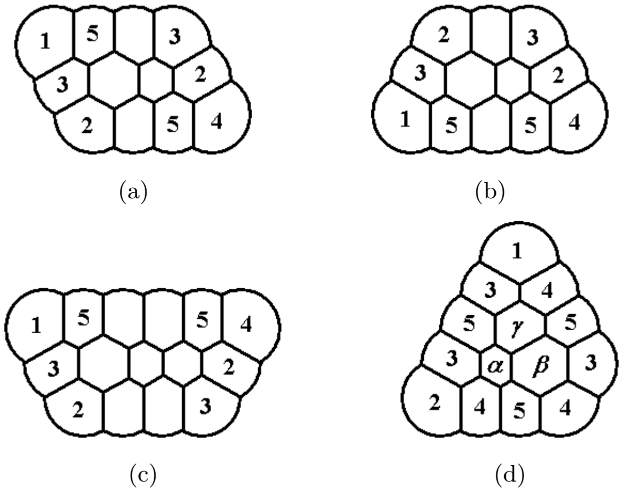

Figure 18: Some 5-bubble which has two or three hexagonal components.

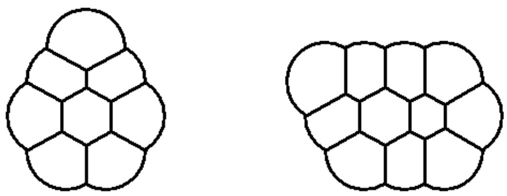

Figure 19: Some 5-bubble which has one or two hexagonal components.

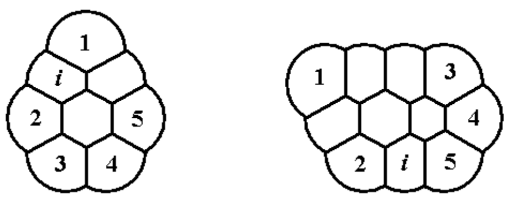

Figure 20: Some 5-bubble which has one or two hexagonal components.

in Figure 21. Now, every 3-sided and4-sided component has different labels. We label them in Figure 22. By Corollary 2.12 and Lemma 3.9, we label them in Figure 23 which is a contradiction with Lemma 3.11.

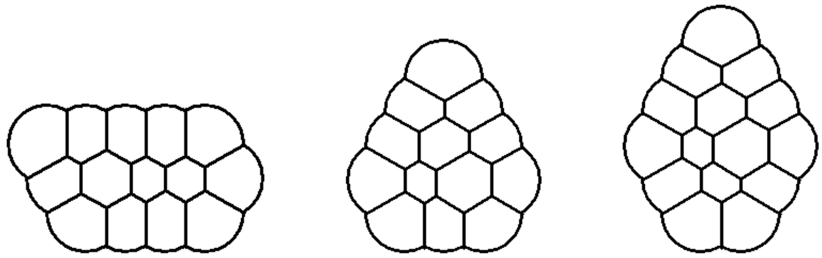

Figure 21: Some 5-bubble which has one or two hexagonal components.

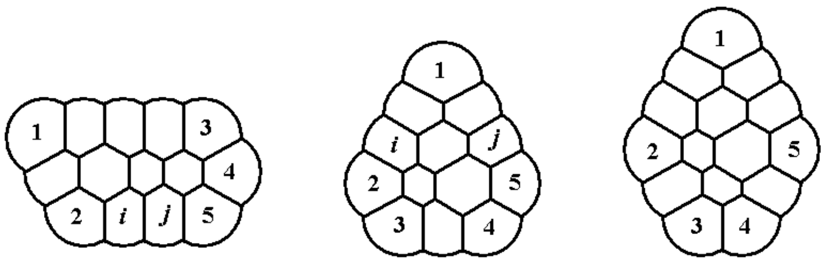

Figure 22: Some 5-bubble which has one or two hexagonal components.

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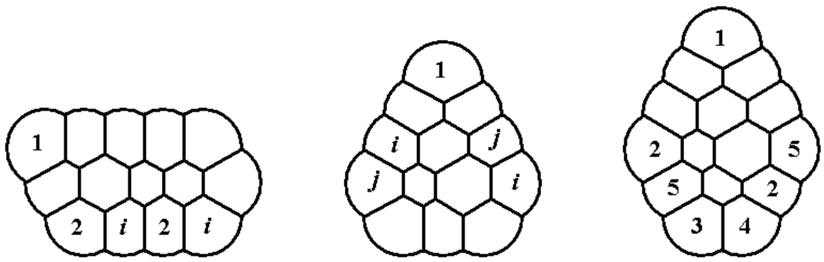

Figure 23: Some 5-bubble which has one or two hexagonal components.

Theorem 3.14. *Let $m \in \{4, 5\}$. A minimizing m-bubble with equal pressures is standard.*

Proof. This follows from Theorem 2.6, Proposition 3.12 and Proposition 3.13. □

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