Thai Journal of Mathematics
Volume 18 Number 2 (2020)
Pages 593–596

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Extention of Bernstein's Type Inequality to Polar Derivative of a Polynomial

Irshad Ahmad[1,*], Abdul Liman[1] and Ruqia Mohammad[2]

[1] *Department of Mathematics, National Institute of Technology, Kashmir, India-190006*
e-mail : irshadmaths84@rediffmail.com (I. Ahmad); abliman22@yahoo.com (A. Liman)

[2] *Department of Mathematics, SSM College of Engineering, Patan Baramulla, Kashmir, India-190006*
e-mail : shafatruqia@gmail.com (R. Mohammad)

Abstract Let $P(z)$ be a polynomial of degree n and $D_\alpha P(z)$ denotes the polar derivative of $P(z)$. Using recently developed interpolation formulation, we obtain an interesting extension of refinement of well known inequality of S. Bernstien for polynomials.

MSC: 30C10; 30C15
Keywords: polynomials; inequalities in the complex domain; polar derivative; zeros

Submission date: 26.08.2018 / Acceptance date: 04.12.2018

1. Introduction and Main Result

Let P_n be the linear space of polynomial of degree at most n and $P \in P_n$. Then concerning the estimate of the maximum of $|P^{'}(z)|$ on the unit circle $|z| = 1$, we have

$$\max_{|z|=1} |P^{'}(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Inequality (1.1) is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial (for reference see [1]). In (1.1) equality holds only for $P(z) = \alpha z^n, \ |\alpha| \neq 0$, that is, if and only if $P(z)$ has all zeros at the origin. Recently Frappier, Rahman, and Ruscheweyh [2, Theorem 8] that if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P^{'}(z)| \leq n \max_{1\leq k\leq 2n} |P(e^{\frac{ik\pi}{n}})|. \tag{1.2}$$

Clearly (1.2) represents a refinement of (1.1), since the maximum of $|P(z)|$ on $|z| = 1$, may be larger than the maximum of $|P(z)|$ taken over the $(2n)$th roots of unity, as is shown by the simple example $|P(z)| = z^n + ia$, $a > 0$. Aziz [3] in this direction produced the following result:

*Corresponding author.

Published by The Mathematical Association of Thailand.

Theorem 1.1. [3] *If $P(z)$ is a polynomial of degree n, then for every given real λ,*

$$\max_{|z|=1}|P^{'}(z)| \leq \frac{n}{2}\ [M_\lambda + M_{\lambda+\pi}], \tag{1.3}$$

where

$$M_\lambda = \max_{1\leq k\leq n}|P(e^{\frac{i(2k\pi+\lambda)}{n}})|,$$

and $M_{\lambda+\pi}$ is obtained by replacing α by $\lambda+\pi$ from definition.
The result is the best possible and equality in (1.3) holds for $p(z) = z^n + re^{i\lambda}, -1 \leq r \leq 1$.

Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to the point α, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P^{'}(z).$$

The polynomial $D_\alpha P(z)$ is of degree atmost $n-1$ and it generalizes the ordinary derivative, where α is real or complex number.

In this paper, we extend (1.3) to the polar derivative of a polynomial $P(z)$. In fact, we prove

Theorem 1.2. *If $P(z)$ is a polynomial of degree n, then for every $\lambda \in \mathbb{R}$ (field of real numbers) and $\alpha \in \mathbb{C}$ (field of complex numbers),*

$$|D_\alpha P(z) - nP(z)| \leq \frac{n}{2}\ |\alpha - z|[M_\lambda + M_{\lambda+\pi}], \tag{1.4}$$

where

$$M_\lambda = \max_{1\leq k\leq n}|P(e^{\frac{i(2k\pi+\lambda)}{n}})|$$

and $M_{\lambda+\pi}$ is obtained by replacing λ by $\lambda+\pi$ from definition.
The result is best possible and equality in (1.4) holds for $P(z) = z^n + re^{i\lambda}, -1 \leq r \leq 1$.

Remark 1.3. If we divide on both sides by α and letting α to infinity, we get Theorem 1.1.

Taking $\lambda = 0$ in Theorem 1.2, we obtain

Corollary 1.4. *If $P(z)$ is a polynomial of degree n, then*

$$|D_\alpha P(z) - nP(z)| \leq \frac{n}{2}|\alpha - z|[\max_{1\leq k\leq n}|P(e^{\frac{i(2k\pi)}{n}})| + \max_{1\leq k\leq n}|P(e^{\frac{i(2k+1)\pi}{n}})|].$$

The result is the best possible and equality holds for $p(z) = z^n - r, -1 \leq r \leq 1$.

For the proof of the Theorem 1.2, we need the following lemma, which is an interpolation formula due to author [4].

Lemma 1.5. *If $P(z)$ is a polynomial of degree n and $z_1, z_2, ..., z_n$ are the zeros of $z^n + a$, where a is any non-zero complex number, then for every $t \in C$ such that $t^n + a \neq 0$, we have*

$$P^{'}(t) = \frac{nt^{n-1}}{t^n + a}P(t) + \frac{t^n + a}{na}\sum_{k=1}^{n} P(z_k)\frac{z_k}{(z_k - t)^2} \tag{1.5}$$

and

$$\frac{1}{na}\sum_{k=1}^{n}\frac{z_k t}{(z_k-t)^2} = -\frac{nt^n}{(t^n+a)^2}. \tag{1.6}$$

2. Proof of Theorem 1.2

Let $P(z)$ be a polynomial of degree n, therefore by definition of polar derivative we have,

$$D_\alpha P(z) = nP(z) + (\alpha - z)P^{'}(z).$$

Equivalently

$$D_\alpha P(t) = nP(t) + (\alpha - t)P^{'}(t).$$

In Lemma 1.5, we take $a = e^{i\beta}$, where β is an arbitrary real number. Then $z_1, z_2, ..., z_n$ are zeros of $z^n = e^{i\beta}$, that is, these points lie on the unit circle, therefore for every complex number t, $|t| = 1$ and $t^n + a \neq 0$ so that $t \neq z_k, k = 1, 2, ..., n$. By (1.5), we have

$$D_\alpha P(t) = nP(t) + (\alpha - t)\left[\frac{nt^{n-1}}{t^n+a}P(t) + \frac{t^n+a}{na}\sum_{k=1}^{n}P(z_k)\frac{z_k}{(z_k-t)^2}\right].$$

$$\left|D_\alpha P(t) - nP(t)\left[1 + \frac{(\alpha-t)t^{n-1}}{t^n+a}\right]\right| = \left|\frac{(\alpha-t)(t^n+a)}{na}\sum_{k=1}^{n}P(z_k)\frac{z_k}{(z_k-t)^2}\right|$$

$$\left|(t^n+a)D_\alpha P(t) - nP(t)(\alpha t^{n-1}+a)\right| \leq |(\alpha-t)|\left|\frac{(a+t^n)^2}{na}\right|\sum_{k=1}^{n}\left|P(z_k)\frac{z_k}{(z_k-t)^2}\right|. \tag{2.1}$$

Now if $|t| = 1$, $|z_k| = 1, t \neq z_k$, then it can be easily verified that $\frac{z_k}{(z_k-t)^2}$ is a negative real number. Further for $|a| = 1, |t| = 1$, and $a + t^n \neq 0$, it can be easily verified that $\frac{(a+t^n)^2}{at^n}$ is a positive real number. Now using these facts and (1.6), we have

$$\left|\frac{a+t^n}{nat^n}\right|\sum_{k=1}^{n}\left|\frac{z_k t}{(z_k-t)^2}\right| = \frac{a+t^n}{nat^n}\sum_{k=1}^{n}\left[-\frac{z_k t}{(z_k-t)^2}\right] = n. \tag{2.2}$$

From (2.1), we have

$$\begin{aligned}\left|(t^n+a)D_\alpha P(t) - nP(t)(\alpha t^{n-1}+a)\right| &\leq |(\alpha-t)|\left|\frac{(a+t^n)^2}{nat^n}\right|\sum_{k=1}^{n}\frac{z_k t}{(z_k-t)^2}\left[\max_{1\leq k\leq n}|P(z_k)|\right]\\ &= n\left|(\alpha-t)\right|\max_{1\leq k\leq n}|P(z_k)|.\end{aligned}$$

Which on simplification gives

$$\left|t^{n-1}\left[n\alpha P(t) - tD_\alpha P(t)\right] - a\left[D_\alpha P(t) - nP(t)\right]\right| \leq n\left|(\alpha-t)\right|\max_{1\leq k\leq n}|P(z_k)|. \tag{2.3}$$

Inequality (2.3) is obviously true for $t = z_k, k = 1, 2, ..., n$. We conclude that for every real β, we have for $|t| = 1$ from (2.3) that

$$\left|t^{n-1}\left[n\alpha P(t)-tD_\alpha P(t)\right]-e^{i\beta}\left[D_\alpha P(t)-nP(t)\right]\right|\leq n\left|(\alpha-t)\right|\max_{1\leq k\leq n}\left|P(z_k)\right|. \tag{2.4}$$

Now $z_1, z_2, ..., z_n$ are zeros of $z^n + e^{i\beta}$, then

$$z_k = e^{\frac{i((2k+1)\pi+\beta)}{n}}.$$

Now using value of z_k in (2.4) and also replace β by λ and next β by $\lambda+\pi$, where λ is real, we get for $|z|=1$

$$\left|z^{n-1}\left[n\alpha P(z)-zD_\alpha P(z)\right]-e^{i\lambda}\left[D_\alpha P(z)-nP(z)\right]\right|\leq n\left|(\alpha-z)\right|\max_{1\leq k\leq n}\left|P(e^{\frac{i((2k+1)\pi+\lambda)}{n}})\right| \tag{2.5}$$

and

$$\left|z^{n-1}\left[n\alpha P(z)-zD_\alpha P(z)\right]+e^{i\lambda}\left[D_\alpha P(z)-nP(z)\right]\right|\leq n\left|(\alpha-z)\right|\max_{1\leq k\leq n}\left|P(e^{\frac{i(2k\pi+\lambda)}{n}})\right|. \tag{2.6}$$

Now

$$\begin{aligned}2\left|D_\alpha P(z)-nP(z)\right| &= \left|e^{i\lambda}\left[D_\alpha P(z)-nP(z)\right]+e^{i\lambda}\left[D_\alpha P(z)-nP(z)\right]\right|\\ &\leq \left|e^{i\lambda}\left[D_\alpha P(z)-nP(z)\right]-z^{n-1}\left[n\alpha P(z)-zD_\alpha P(z)\right]\right|+\left|e^{i\lambda}\left[D_\alpha P(z)-nP(z)\right]\right.\\ &\qquad \left.+z^{n-1}\left[n\alpha P(z)-zD_\alpha P(z)\right]\right|.\end{aligned}$$

Using (2.5) and (2.6), we get

$$2\left|D_\alpha P(z)-nP(z)\right|\leq n\left|(\alpha-z)\right|\left[M_\lambda+M_{\lambda+\pi}\right].$$

Hence

$$\max_{|z|=1}\left|D_\alpha P(z)-nP(z)\right|\leq\frac{n}{2}\left|(\alpha-z)\right|\left[M_\lambda+M_{\lambda+\pi}\right].$$

This proves the theorem completely.

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript.

References

[1] A.C. Schaeffer, Inequalities of A. Markoff and S. Bernstein for polynomials and related functions, Bull. Amer. Math. Soc. 47 (1941) 565–579.

[2] C. Frappier, Q.I. Rahman, St. Ruscheweyh, New inequalities for polynomials, Trans. Amer. Math. Soc. 288 (1985) 69–99.

[3] A. Aziz, A refinement of an inequality of S. Bernstein, Journal of Matematical Analysis and Applications 144 (1989) 226–235.

[4] A. Aziz, Inequalities for the polynomial with a prescribed zero, J. Approx. Theory 41 (1984) 15–20.