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The Stability of an Additive-Quartic Functional Equation in Quasi-β-Normed Spaces with the Fixed Point Alternative

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Abstract The aim of this paper is to use the fixed point alternative for investigating the generalized Hyers-Ulam stability for the following additive-quartic functional equation

$$f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x)+24f(y)=13[f(x+y)+f(x-y)]+12f(2y),$$

where f maps from a normed space to a quasi-β-Banach space.

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1. Introduction

In 1940, Ulam [1] proposed the following classical stability problem.

- Let f be a mapping from a group $(G_1, \bullet)$ to a metric group $(G_2, *)$ with the metric d such that

  $$d(f(x \bullet y), f(x) * f(y)) \leq \epsilon,$$

  for all $x, y \in G_1$, where $\epsilon > 0$. Do there exist a unique homomorphism $H : G_1 \to G_2$ and a constant $\delta > 0$ such that

  $$d(f(x), H(x)) \leq \delta,$$

  for all $x \in G_1$?

Next year, Hyers [2] solved this problem under the assumption that the function f maps between two Banach spaces. In 1978, a generalization of Hyers' result was obtained by Rassias [3] for a mapping f which maps from a Banach space X to a Banach space Y by considering an unbounded Cauchy difference

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \left(\|x\|^p + \|y\|^p\right),$$

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for all $x, y \in X$, where $\epsilon > 0$ and $0 < p < 1$. In 1994, a generalized Hyers-Ulam stability of Rassias' theorem was proved by Gãvruta [4] by replacing the unbounded Cauchy difference with a general control function.

Nowadays, several new investigations on the various functional equations have been suggested by many authors. Here, we give examples of the study in this way. In 1999, Rassias [5] introduced the functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4[f(x+y) + f(x-y) + 6f(y)], \tag{1.1}$$

where f maps from a vector space X into a real vector space Y, which is called quartic functional equation. He also call every solution of (1.1) as a quartic function. Furthermore, he proved the Hyers-Ulam stability problem for the functional equation (1.1), where X is a normed space and Y is a real Banach space. The quartic functional equation was employed by other authors. In 2003, Chung and Sahoo [6] proved the general solution of the quartic functional equation (1.1), where f maps from $\mathbb{R}$ to $\mathbb{R}$.

In 2004, Sahoo [7] solved the general solution of the following functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4[f(x+y) + f(x-y)], \tag{1.2}$$

where f maps from $\mathbb{R}$ to $\mathbb{R}$.

In 2008, Petapirak and Nakmahachalasint [8] had shown that the function f maps between vector spaces X and Y satisfying the functional equation

$$f(3x+y) + f(x+3y) = 64f(x) + 64f(y) + 24f(x+y) - 6f(x-y), \tag{1.3}$$

for all $x, y \in X$ if and only if there exists a 4-additive symmetric function $A : X^4 \to Y$ such that $f(x) = A(x, x, x, x)$ for all $x \in X$. They also investigated the generalized Hyers-Ulam stability of the functional equation (1.3).

In 2010, Gordji [9] proved that the function $f : X \to Y$, where X and Y are vector spaces, satisfies the functional equation

$$f(2x+y) + f(2x-y) = 4\left[f(x+y) + f(x-y)\right] - \frac{3}{7}[f(2y) - 2f(y)] + 2f(2x) - 8f(x), \tag{1.4}$$

for all $x, y \in X$ if and only if there exist a unique symmetric multiadditive function $B : X^4 \to Y$ and a unique additive function $A : X \to Y$ such that $f(x) = B(x, x, x, x) + A(x)$ for all $x \in X$ and he considered the generalized Hyers-Ulam stability of (1.4), where f maps from a real normed space X to a real Banach space Y.

In 2013, Hengkrawit and Thanyacharoen [10] considered the following functional equation

$$\begin{aligned} f(x+3y) + f(x-3y) + f(x+2y) + f(x-2y) + 22f(x) \\ = 13\left[f(x+y) + f(x-y)\right] + 168f(y), \end{aligned} \tag{1.5}$$

where f maps from $\mathbb{R}$ to $\mathbb{R}$. Its stability is investigated and they solved that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.5) if and only if it is of the form $f(x) = A(x, x, x, x)$, where $A : \mathbb{R}^4 \to \mathbb{R}$ is the diagonal of 4-additive symmetric function.

In 2015, Hengkrawit and Thanyacharoen [11] determined the general solution of the generalized additive-quartic functional equation

$$\begin{aligned} f(x+3y) + f(x-3y) + f(x+2y) + f(x-2y) + 22f(x) + 24f(y) \\ = 13[f(x+y) + f(x-y)] + 12f(2y), \end{aligned} \tag{1.6}$$

where f maps from $\mathbb{R}$ to $\mathbb{R}$, and studied the Hyers-Ulam stability of this functional equation.

The main goal of this paper is to use the fixed point alternative for proving the generalized Hyers-Ulam stability for the functional equation

$$f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x)+24f(y) \\ = 13[f(x+y)+f(x-y)]+12f(2y), \tag{1.7}$$

where f maps from a normed space to a quasi-β-Banach space.

2. Preliminaries

In this section, we will recall some basic concepts of a quasi-β-normed space and the important tools from the fixed point theory for proving the main result.

Definition 2.1 ([12])**.** Let β be a real number with $0<\beta\le 1$, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and X be a vector space over $\mathbb{K}$. A function $\|\cdot\|: X\to\mathbb{R}$ is called a *quasi-β-norm* if it satisfies the following conditions:

(1) $\|x\|=0$ if and only if $x=0$;
(2) $\|rx\|=|r|^{\beta}\|x\|$ for all $r\in\mathbb{K}$ and all $x\in X$;
(3) there is a constant $K\ge 1$ such that

$$\|x+y\|\le K\left(\|x\|+\|y\|\right),$$

for all $x,y\in X$.

Also, the pair $(X,\|\cdot\|)$ or $(X,\|\cdot\|,K)$ is called a *quasi-β-normed space.* The smallest possible K is called the *modulus of concavity of* $\|\cdot\|$.

Remark 2.2. In a quasi-β-normed space $(X,\|\cdot\|)$, we have $\|x\|\ge 0$ for all $x\in X$.

Definition 2.3 ([12])**.** A quasi-β-normed space $(X,\|\cdot\|,K)$ is called a (β,p)*-normed space* if there exists a real number $p\in(0,1]$ such that

$$\|x+y\|^{p}\le\|x\|^{p}+\|y\|^{p},$$

for all $x,y\in X$. In this case, $\|\cdot\|$ is also called a $(\beta,p)-$*norm* on X.

We can refer to [12] for the more details in quasi-β-normed spaces.

Next, we give the one of fundamental results due to Diaz and Margolis [13] in the fixed point theory which is the main tool for investigating many stability results.

Theorem 2.4 ([13])**.** *Let (X,d) be a complete generalized metric space and $J: X\to X$ be a strictly contractive mapping with some Lipschitz constant L with $0\le L<1$. Then for each given element $x\in X$, either*

$$d(J^n x, J^{n+1}x)=\infty,$$

for all nonnegative integers n or there exists a positive integer n_0 such that the following assertions hold:

(1) $d(J^n x,J^{n+1}x)<\infty$ for all $n\ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point $y*$ of J;
(3) y^* is the unique fixed point of J in the set $Y:=\{y\in X|d(J^{n_0}x,y)<\infty\}$;
(4) $d(y,y^*)\le\frac{1}{1-L}d(y,Jy)$ for all $y\in Y$.

In 2007, a generalization of Theorem 2.4 was proved by Aydi and Czerwik [14] in generalized b-metric spaces.

Theorem 2.5 ([14]). *Let (X, D, K) be a complete generalized b-metric space and $T : X \to X$ satisfies the condition*

$$D(T(x), T(y)) \leq \varphi(D(x, y)),$$

for all $x, y \in X$ and $D(x, y) < \infty$, where $\varphi : [0, \infty) \to [0, \infty)$ is nondecreasing and

$$\lim_{n\to\infty} \varphi^n(z) = 0,$$

for $z > 0$. Then for each given element $x \in X$, either

$$D(T^n x, T^{n+1}x) = \infty,$$

for all nonnegative integers n or there exists a positive integer k such that the following assertions hold:

(1) $D(T^k x, T^{k+1}x) < \infty$;
(2) the sequence $\{T^n x\}$ converges to a fixed point u of T;
(3) u is the unique fixed point of T in the set $B := \{y \in X | D(T^k x, y) < \infty\}$.

Remark 2.6. In Theorem 2.5, if $K = 1$ and a function $\varphi : [0, \infty) \to [0, \infty)$ is defined by

$$\varphi(t) = Lt,$$

for all $t \in [0, \infty)$, where $L \in [0, 1)$, then this theorem reduces to Theorem 2.4.

Remark 2.7. In Theorem 2.5, if u is a fixed point of T and a function $\varphi : [0, \infty) \to [0, \infty)$ defined by

$$\varphi(t) = Lt,$$

for all $t \in [0, \infty)$, where $L \in [0, 1)$ with $KL < 1$, then for each $y \in X$, we have

$$\begin{aligned} D(u, y) &\leq K\left[D(u, Ty) + D(Ty, y)\right] \\ &= K\left[D(Tu, Ty) + D(Ty, y)\right] \\ &= K\left[LD(u, y) + D(Ty, y)\right]. \end{aligned}$$

This implies that

$$D(u, y) \leq \left(\frac{K}{1 - KL}\right) D(Ty, y),$$

for all $y \in X$.

Elegancy of the above fixed point result fascinates several mathematicians. Subsequently, stability results for several functional equations having the full force of such fixed point result were obtained.

3. Main Results

Throughout this section, let X be a normed space, Y be a $(\beta, p)-$Banach space and $f : X \to Y$ be a mapping. For each $x, y \in X$, we will use the following symbol:

$$\begin{aligned} Df(x, y) :=& f(x + 3y) + f(x - 3y) + f(x + 2y) + f(x - 2y) + 22f(x) \\ & - 13f(x + y) - 13f(x - y) + 24f(y) - 12f(2y). \end{aligned}$$

First, we give an auxiliary lemmas.

Lemma 3.1. *Let X be a normed space, Y be a $(\beta,p)-$Banach space and $f: X \to Y$ be a mapping. If f satisfies (1.7), then f is of the form*

$$\begin{aligned} &21f(x+2y)+21f(x-2y)-84f(x+y)-84f(x-y)+126f(x) \\ &\qquad +70f(y)-30f(-y)-33f(2y)+15f(-2y)-f(4y)=0, \end{aligned} \tag{3.1}$$

for all $x,y \in X$.

Proof. Suppose that f satisfies (1.7). Substituting x by $x+3y$ into (1.7), we get

$$\begin{aligned} &f(x+6y)+f(x)+f(x+5y)+f(x+y)+22f(x+3y)-13f(x+4y) \\ &\qquad -13f(x+2y)+24f(y)-12f(2y)=0, \end{aligned} \tag{3.2}$$

for all $x,y \in X$. Substituting x by $x+2y$ into (1.7), we get

$$\begin{aligned} &f(x+5y)+f(x-y)+f(x+4y)+f(x)+22f(x+2y)-13f(x+3y) \\ &\qquad -13f(x+y)+24f(y)-12f(2y)=0, \end{aligned} \tag{3.3}$$

for all $x,y \in X$. From (3.2) and (3.3), we obtain

$$\begin{aligned} &f(x+6y)-14f(x+4y)+35f(x+3y)-35f(x+2y) \\ &\qquad +14f(x+y)-f(x-y)=0, \end{aligned} \tag{3.4}$$

for all $x,y \in X$. Substituting y by $-y$ into (3.4), we get

$$\begin{aligned} &f(x-6y)-14f(x-4y)+35f(x-3y)-35f(x-2y) \\ &\qquad +14f(x-y)-f(x+y)=0, \end{aligned} \tag{3.5}$$

for all $x,y \in X$. From (3.4) and (3.5), we obtain

$$\begin{aligned} &f(x+6y)+f(x-6y)-14f(x+4y)-14f(x-4y)+35f(x+3y)+35f(x-3y) \\ &\qquad -35f(x+2y)-35f(x-2y)+13f(x+y)+13f(x-y)=0, \end{aligned} \tag{3.6}$$

for all $x,y \in X$. Substituting y by $2y$ into (1.7), we have

$$\begin{aligned} &f(x+6y)+f(x-6y)+f(x+4y)+f(x-4y)+22f(x)-13f(x+2y) \\ &\qquad -13f(x-2y)+24f(2y)-12f(4y)=0, \end{aligned} \tag{3.7}$$

for all $x,y \in X$. From (3.6) and (3.7), we obtain

$$\begin{aligned} &15f(x+4y)+15f(x-4y)-35f(x+3y)-35f(x-3y)+22f(x+2y) \\ &\quad +22f(x-2y)-13f(x+y)-13f(x-y)+22f(x)+24f(2y)-12f(4y)=0, \end{aligned} \tag{3.8}$$

for all $x,y \in X$. Substituting x by $x+y$ into (1.7), we get

$$\begin{aligned} &f(x+4y)+f(x-2y)+f(x+3y)+f(x-y)+22f(x+y)-13f(x+2y) \\ &\qquad -13f(x)+24f(y)-12f(2y)=0, \end{aligned} \tag{3.9}$$

for all $x,y \in X$. Substituting y by $-y$ into (3.9), we have

$$\begin{aligned} &f(x-4y)+f(x+2y)+f(x-3y)+f(x+y)+22f(x-y) \\ &\qquad -13f(x-2y)-13f(x)+24f(-y)-12f(-2y)=0, \end{aligned} \tag{3.10}$$

for all $x, y \in X$. From (3.9) and (3.10), we obtain

$$\begin{aligned}&f(x+4y)+f(x-4y)+f(x+3y)+f(x-3y)-12f(x+2y)-12f(x-2y)\\&+23f(x+y)+23f(x-y)-26f(x)+24f(y)-12f(2y)+24f(-y)\\&-12f(-2y)=0,\end{aligned} \tag{3.11}$$

thus

$$\begin{aligned}&15f(x+4y)+15f(x-4y)+15f(x+3y)+15f(x-3y)-180f(x+2y)\\&-180f(x-2y)+345f(x+y)+345f(x-y)-390f(x)+360f(y)+360f(-y)\\&-180f(2y)-180f(-2y)=0,\end{aligned} \tag{3.12}$$

for all $x, y \in X$. From (3.8) and (3.12), we obtain

$$\begin{aligned}&50f(x+3y)+50f(x-3y)-202f(x+2y)-202f(x-2y)+358f(x+y)\\&+358f(x-y)-412f(x)+360f(y)+360f(-y)-204f(2y)\\&-180f(-2y)+12f(4y)=0,\end{aligned} \tag{3.13}$$

for all $x, y \in X$. From (1.7), we get

$$\begin{aligned}&50f(x+3y)+50f(x-3y)+50f(x+2y)+50f(x-2y)-650f(x+y)\\&\qquad -650f(x-y)+1100f(x)+1200f(y)-600f(2y)=0,\end{aligned} \tag{3.14}$$

for all $x, y \in X$. From (3.13) and (3.14), we have

$$\begin{aligned}&252f(x+2y)+252f(x-2y)-1008f(x+y)-1008f(x-y)+1512f(x)\\&\qquad +840f(y)-360f(-y)-396f(2y)+180f(-2y)-12f(4y)=0,\end{aligned} \tag{3.15}$$

thus

$$\begin{aligned}&21f(x+2y)+21f(x-2y)-84f(x+y)-84f(x-y)+126f(x)\\&\qquad +70f(y)-30f(-y)-33f(2y)+15f(-2y)-f(4y)=0,\end{aligned} \tag{3.16}$$

for all $x, y \in X$. ■

Lemma 3.2. *Let X be a normed space, Y be a (β, p)-Banach space and $f : X \to Y$ be a mapping satisfying* (1.7). *Then the following assertions hold:*

(1) f is even if and only if f is quartic;
(2) f is odd if and only if f is additive.

Proof. Replacing x and y by 0 in (1.7), we have $f(0) = 0$.

(1) It is easy to see that if f is quartic, then f is even (see in [5]). Next, we will show that if f is even, then f is quartic. Suppose that f is even. By Lemma 3.1, f is of the form (3.1). Putting $x = 0$ in (3.1), since f is even and $f(0) = 0$, we obtain

$$24f(2y)-128f(y)-f(4y)=0, \tag{3.17}$$

for all $y \in X$. It follows the proof of Lemma 3.1 that

$$\begin{aligned}&15f(x+4y)+15f(x-4y)-35f(x+3y)-35f(x-3y)+22f(x+2y)\\&\qquad +22f(x-2y)-13f(x+y)-13f(x-y)+22f(x)+24f(2y)\\&\qquad -12f(4y)=0,\end{aligned} \tag{3.18}$$

for all $x, y \in X$. Putting $x = 0$ in (3.18), since f is even and $f(0) = 0$, we obtain

$$30f(4y) - 70f(3y) + 44f(2y) - 26f(y) + 24f(2y) - 12f(4y) = 0,$$

thus

$$9f(4y) - 35f(3y) + 34f(2y) - 13f(y) = 0, \tag{3.19}$$

for all $y \in X$. From (3.17), we get

$$216f(2y) - 1152f(y) - 9f(4y) = 0, \tag{3.20}$$

for all $y \in X$. From (3.19) and (3.20), we have

$$-35f(3y) + 250f(2y) - 1165f(y) = 0, \tag{3.21}$$

for all $y \in X$. Letting $x = 0$ in (1.7), since f is even and $f(0) = 0$, we obtain

$$f(3y) - 5f(2y) - f(y) = 0,$$

thus

$$35f(3y) - 175f(2y) - 35f(y) = 0, \tag{3.22}$$

for all $y \in X$. From (3.21) and (3.22), we get

$$f(2y) = 16f(y), \tag{3.23}$$

for all $y \in X$. From (3.1), we have

$$f(x + 2y) + f(x - 2y) - 4f(x + y) - 4f(x - y) + 6f(x) - 24f(y) = 0, \tag{3.24}$$

for all $x, y \in X$. So f is a quartic mapping.

(2) It is easy to see that if f is additive, then f is add. So we must show that if f is odd, then f is additive. Suppose that f is odd. By Lemma 3.1, f is of the form (3.1). Substituting y by $-y$ into (1.7), since f is odd, we get

$$\begin{aligned} f(x + 3y) + f(x - 3y) + f(x + 2y) + f(x - 2y) + 22f(x) - 13f(x + y)& \\ -13f(x - y) - 24f(y) + 12f(2y) = 0,& \end{aligned} \tag{3.25}$$

for all $x, y \in X$. From (1.7) and (3.25), we obtain

$$f(2y) = 2f(y), \tag{3.26}$$

for all $y \in X$. From (3.1) and (3.26), we have

$$f(x + 2y) + f(x - 2y) - 4f(x + y) - 4f(x - y) + 6f(x) = 0, \tag{3.27}$$

for all $x, y \in X$. Substituting x by $2x$ into (3.27) and using (3.26), we get

$$2f(2x + y) + 2f(2x - y) - f(x + y) - f(x - y) - 6f(x) = 0, \tag{3.28}$$

for all $x, y \in X$. Interchanging x into y in (3.27), we have

$$f(2x + y) - f(2x - y) - 4f(x + y) + 4f(x - y) + 6f(y) = 0, \tag{3.29}$$

for all $x, y \in X$. Substituting y by $-y$ into (3.29), we get

$$f(2x - y) - f(2x + y) - 4f(x - y) + 4f(x + y) - 6f(y) = 0, \tag{3.30}$$

for all $x, y \in X$. Replacing x by $2x$ in (3.28) and using (3.26), we get

$$2f(4x + y) + 2f(4x - y) - f(2x + y) - f(2x - y) - 12f(x) = 0, \tag{3.31}$$

for all $x, y \in X$. From (3.28) and (3.31), we obtain

$$4f(4x+y)+4f(4x-y)-f(x+y)-f(x-y)-30f(x)=0, \tag{3.32}$$

for all $x, y \in X$. Substituting y by $y+2x$ into (3.28), we have

$$4f(4x+y)-4f(y)-2f(3x+y)+2f(x+y)-12f(x)=0, \tag{3.33}$$

for all $x, y \in X$. Replacing y by $-y$ in (3.33), we get

$$4f(4x-y)+4f(y)-2f(3x-y)+2f(x-y)-12f(x)=0, \tag{3.34}$$

for all $x, y \in X$. From (3.33) and (3.34), we obtain

$$\begin{aligned}4f(4x+y)+4f(4x-y)-2f(3x+y)-2f(3x-y)+2f(x+y)\\+2f(x-y)-24f(x)=0,\end{aligned} \tag{3.35}$$

for all $x, y \in X$. From (3.32) and (3.35), we obtain

$$2f(3x+y)+2f(3x-y)-3f(x+y)-3f(x-y)-6f(x)=0, \tag{3.36}$$

for all $x, y \in X$. Substituting y by $x-y$ into (3.28), we have

$$2f(3x-y)+2f(x+y)-f(2x-y)-f(y)-6f(x)=0, \tag{3.37}$$

for all $x, y \in X$. Substituting y by $x+y$ into (3.28), we get

$$2f(3x+y)+2f(x-y)-f(2x+y)+f(y)-6f(x)=0, \tag{3.38}$$

for all $x, y \in X$. From (3.37) and (3.38), we obtain

$$\begin{aligned}2f(3x+y)+2f(3x-y)-f(2x+y)-f(2x-y)+2f(x+y)\\+2f(x-y)-12f(x)=0,\end{aligned} \tag{3.39}$$

for all $x, y \in X$. From (3.28) and (3.39), we have

$$4f(3x+y)+4f(3x-y)+3f(x+y)+3f(x-y)-30f(x)=0, \tag{3.40}$$

for all $x, y \in X$. From (3.36) and (3.40), we have

$$f(x+y)+f(x-y)-2f(x)=0, \tag{3.41}$$

for all $x, y \in X$. Interchanging x into y in (3.41), we get

$$f(x+y)-f(x-y)-2f(y)=0, \tag{3.42}$$

for all $x, y \in X$. From (3.41) and (3.42), we obtain

$$f(x+y)=f(x)+f(y), \tag{3.43}$$

for all $x, y \in X$. So f is an additive mapping. ■

Next, we are going to consider the stability of the additive-quartic functional equation (1.7).

Theorem 3.3. *Let X be a normed space, Y be a $(\beta, p)-$Banach space with the modulus of concavity K and $\phi : X \times X \to [0, \infty)$ be a function such that*

$$\phi(2x, 2y) \le L\phi(x, y), \tag{3.44}$$

for all $x, y \in X$, where $0 \le L < 1$ with $KL < 1$. Suppose that $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and

$$\|Df(x, y)\| \le \phi(x, y), \tag{3.45}$$

for all $x, y \in X$*. Then there exist a unique quartic function* $Q : X \to Y$ *and a unique additive function* $A : X \to Y$ *such that*

$$\left\| \frac{f(x) + f(-x)}{2} - Q(x) \right\| \le \left(\frac{11^\beta K^6}{480^\beta - 30^\beta KL} \right) \tilde{\psi}(x), \tag{3.46}$$

$$\left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| \le \left(\frac{K^2}{48^\beta - 24^\beta KL} \right) \tilde{\psi}(x), \tag{3.47}$$

and

$$\|f(x) - Q(x) - A(x)\| \le \left(\frac{11^\beta K^7}{480^\beta - 30^\beta KL} + \frac{K^3}{48^\beta - 24^\beta KL} \right) \tilde{\psi}(x), \tag{3.48}$$

for all $x \in X$*, where*

$$\tilde{\psi}(x) := \psi(3x, x) + \psi(2x, x) + \psi(x, x) + \psi(0, x) + \psi(0, 2x)$$

such that $\psi(x, y) := \phi(x, y) + \phi(-x, -y)$.

Proof. Let $\Omega := \{g : X \to Y | g(0) = 0\}$. Define a generalized b-metric d on Ω by

$$d(g, h) = \inf\{c \in \mathbb{R}^+ | \, \|g(x) - h(x)\| \le c\tilde{\psi}(x) \text{ for all } x \in X\}.$$

Since Y is a $(\beta, p)-$Banach space, (Ω, d) is a generalized complete b-metric space. Let $f_1 : X \to Y$ be the function defined by $f_1(x) := \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then $f_1(0) = 0$ and $f_1(x) = f_1(-x)$ for all $x \in X$. Substituting x by $-x$ and y by $-y$ into (3.45), we get

$$\|Df(-x, -y)\| \le \phi(-x, -y), \tag{3.49}$$

for all $x, y \in X$. From (3.45) and (3.49), we obtain

$$\|Df_1(x, y)\| \le \frac{K}{2^\beta} \psi(x, y), \tag{3.50}$$

for all $x, y \in X$. Putting $x = 3y$ in (3.50) and using the fact that $f_1(0) = 0$, we have

$$\|f_1(6y) + f_1(5y) - 13f_1(4y) + 22f_1(3y) - 25f_1(2y) + 25f_1(y)\| \le \frac{K}{2^\beta} \psi(3y, y), \tag{3.51}$$

for all $y \in X$. Putting $x = 2y$ in (3.50) and using the fact that $f_1(0) = 0$, we get

$$\|f_1(5y) + f_1(4y) - 13f_1(3y) + 10f_1(2y) + 12f_1(y)\| \le \frac{K}{2^\beta} \psi(2y, y), \tag{3.52}$$

for all $y \in X$. From (3.51) and (3.52), we obtain

$$\|f_1(6y) - 14f_1(4y) + 35f_1(3y) - 35f_1(2y) + 13f_1(y)\| \le \frac{K^2}{2^\beta} \left(\psi(3y, y) + \psi(2y, y) \right), \tag{3.53}$$

for all $y \in X$. Putting $x = 0$ in (3.50), we have

$$\|2f_1(3y) - 10f_1(2y) - 2f_1(y)\| \le \frac{K}{2^\beta} \psi(0, y), \tag{3.54}$$

and so

$$\|f_1(3y) - 5f_1(2y) - f_1(y)\| \le \frac{K}{4^\beta} \psi(0, y), \tag{3.55}$$

for all $y \in X$. Replacing y by $2y$ in (3.55), we obtain

$$\|f_1(6y) - 5f_1(4y) - f_1(2y)\| \leq \frac{K}{4^\beta}\psi(0, 2y), \tag{3.56}$$

for all $y \in X$. From (3.53) and (3.56), we obtain

$$\begin{aligned}\|-9f_1(4y) + 35f_1(3y) \ &-34f_1(2y) + 13f_1(y)\| \\ &\leq \frac{K^3}{2^\beta}\left(\psi(3y, y) + \psi(2y, y)\right) + \frac{K^2}{4^\beta}\psi(0, 2y),\end{aligned} \tag{3.57}$$

for all $y \in X$. Putting $x = y$ in (3.50), we have

$$\|f_1(4y) + f_1(3y) - 24f_1(2y) + 47f_1(y)\| \leq \frac{K}{2^\beta}\psi(y, y), \tag{3.58}$$

and so

$$\|9f_1(4y) + 9f_1(3y) - 216f_1(2y) + 423f_1(y)\| \leq \frac{9^\beta K}{2^\beta}\psi(y, y), \tag{3.59}$$

for all $y \in X$. From (3.57) and (3.59), we obtain

$$\begin{aligned}\|44f_1(3y) \ &-250f_1(2y) + 436f_1(y)\| \\ &\leq \frac{K^4}{2^\beta}\left(\psi(3y, y) + \psi(2y, y)\right) + \frac{K^3}{4^\beta}\psi(0, 2y) + \frac{9^\beta K^2}{2^\beta}\psi(y, y),\end{aligned} \tag{3.60}$$

for all $y \in X$. From (3.55), we have

$$\|44f_1(3y) - 220f_1(2y) - 44f_1(y)\| \leq 11^\beta K\psi(0, y), \tag{3.61}$$

for all $y \in X$. From (3.60) and (3.61), we obtain

$$\begin{aligned}\|-30f_1(2y) + 480f_1(y)\| \leq &\frac{K^5}{2^\beta}\left(\psi(3y, y) + \psi(2y, y)\right) + \frac{K^4}{4^\beta}\psi(0, 2y) \\ &+ \frac{9^\beta K^3}{2^\beta}\psi(y, y) + 11^\beta K^2\psi(0, y),\end{aligned} \tag{3.62}$$

for all $y \in X$. Thus, we have

$$\begin{aligned}\left\|f_1(y) - \frac{f_1(2y)}{16}\right\| \leq &\frac{K^5}{960^\beta}\left(\psi(3y, y) + \psi(2y, y)\right) + \frac{K^4}{1920^\beta}\psi(0, 2y) \\ &+ \frac{9^\beta K^3}{960^\beta}\psi(y, y) + \frac{11^\beta}{480^\beta}K^2\psi(0, y) \\ \leq &\frac{11^\beta}{480^\beta}K^5\left(\psi(3y, y) + \psi(2y, y) + \psi(y, y) + \psi(0, y) + \psi(0, 2y)\right) \\ = &\frac{11^\beta}{480^\beta}K^5\tilde{\psi}(y),\end{aligned} \tag{3.63}$$

for all $y \in X$. This implies that

$$d(J_e f_1, f_1) \leq \frac{11^\beta}{480^\beta}K^5. \tag{3.64}$$

Define a mapping $J_e : \Omega \to \Omega$ by

$$(J_e g)(x) = 2^{-4}g(2x),$$

for all $x \in X$ and for all $g \in \Omega$. We want to show that

$$d(J_e g, J_e h) \leq \frac{L}{2^{4\beta}} d(g,h),$$

for all $g, h \in \Omega$. Let $g, h \in \Omega$. If $d(g,h) = \infty$ for all $g, h \in \Omega$, then the above inequality is true. So we may assume that $d(g,h) < \infty$. Assume that

$$C_e := \{c \in \mathbb{R}^+ | \|g(x) - h(x)\| \leq c\tilde{\psi}(x) \text{ for all } x \in X\}. \tag{3.65}$$

Since $d(g,h) < \infty$, we obtain $C_e \neq \emptyset$. Suppose that $c \in C_e$. For each $x \in X$, we have

$$\begin{aligned}
\|(J_e g)(x) - (J_e h)(x)\| &= \left\|2^{-4} g(2x) - 2^{-4} h(2x)\right\| \\
&= |2|^{-4\beta} \|g(2x) - h(2x)\| \\
&\leq \frac{c}{2^{4\beta}} \tilde{\psi}(2x) \\
&\leq \frac{cL}{2^{4\beta}} \tilde{\psi}(x),
\end{aligned}$$

and so

$$d(J_e g, J_e h) \leq \frac{cL}{2^{4\beta}}.$$

By taking the infimum on $c \in C_e$, we obtain

$$d(J_e g, J_e h) \leq \frac{L}{2^{4\beta}} d(g,h).$$

Therefore, we can conclude that

$$d(J_e g, J_e h) \leq \frac{L}{2^{4\beta}} d(g,h).$$

for all $g, h \in \Omega$. By taking a function $\varphi : [0,\infty) \to [0,\infty)$ in Theorem 2.5 by

$$\varphi(t) = \frac{L}{2^{4\beta}} t,$$

for all $t \in [0,\infty)$, there is the unique fixed point Q of J_e in Ω such that $\{J_e^n f_1\}$ converges to Q in (Ω, d). By Remark 2.7 and (3.64), we get

$$\begin{aligned}
d(Q, f_1) &\leq \left(\frac{K}{1 - 2^{-4\beta} KL}\right) d(J_e f_1, f_1) \\
&\leq \frac{11^\beta K^6}{480^\beta - 30^\beta KL}.
\end{aligned} \tag{3.66}$$

By (3.45), we have

$$\begin{aligned}
\left\|\frac{Df_1(2^n x, 2^n y)}{2^{4n}}\right\|^p &= \frac{1}{2^{4n\beta p}} \left\|\frac{Df(2^n x, 2^n y) + Df(-2^n x, -2^n y)}{2}\right\|^p \\
&\leq \frac{1}{2^{\beta p + 4n\beta p}} \left(\|Df(2^n x, 2^n y)\|^p + \|Df(-2^n x, -2^n y)\|^p\right) \\
&\leq \frac{1}{2^{\beta p + 4n\beta p}} \left(\phi^p(2^n x, 2^n y) + \phi^p(-2^n x, -2^n y)\right) \\
&\leq \frac{1}{2^{\beta p + 4n\beta p}} L^{np} \left(\phi^p(x,y) + \phi^p(-x,-y)\right) \\
&= \frac{1}{2^{\beta p}} \left(\frac{L}{2^{4\beta}}\right)^{np} \left(\phi^p(x,y) + \phi^p(-x,-y)\right),
\end{aligned}$$

for all $x, y \in X$. Letting $n \to \infty$ in the last inequality, we have

$$DQ(x, y) = 0, \tag{3.67}$$

for all $x, y \in X$. By using the fact that $f_1(0) = 0$, we obtain $Q(0) = 0$. Since f_1 is even, it yields that

$$\begin{aligned}\|Q(x) - Q(-x)\| &= \left\|Q(x) - \lim_{n\to\infty} J_e^n f_1(-x)\right\| \\ &= \left\|Q(x) - \lim_{n\to\infty} J_e^n f_1(x)\right\| \\ &= 0,\end{aligned}$$

for all $x \in X$ and so Q is even. By Lemma 3.2, we have Q is a quartic mapping. From (3.66) we get

$$\left\|\frac{f(x)+f(-x)}{2} - Q(x)\right\| \leq \left(\frac{11^\beta K^6}{480^\beta - 30^\beta KL}\right)\tilde{\psi}(x), \tag{3.68}$$

for all $x \in X$. Let $f_2 : X \to Y$ be the function defined by $f_2(x) := \frac{f(x)-f(-x)}{2}$ for all $x \in X$. Then $f_2(0) = 0$ and $f_2(-x) = -f_2(x)$ for all $x \in X$. From (3.45) we obtain

$$\|Df_2(x, y)\| \leq \frac{K}{2^\beta}\psi(x, y), \tag{3.69}$$

for all $x, y \in X$. Putting $x = 0$ in (3.69), and using the facts that f_2 is odd and $f_2(0) = 0$, we have

$$\|24f_2(y) - 12f_2(2y)\| \leq \frac{K}{2^\beta}\psi(0, y), \tag{3.70}$$

and so

$$\begin{aligned}\left\|f_2(y) - \frac{f_2(2y)}{2}\right\| &\leq \frac{K}{24^\beta 2^\beta}\psi(0, y) \\ &\leq \frac{K}{48^\beta}\left(\psi(3y, y) + \psi(2y, y) + \psi(y, y) + \psi(0, y) + \psi(0, 2y)\right) \\ &= \frac{K}{48^\beta}\tilde{\psi}(y),\end{aligned} \tag{3.71}$$

for all $y \in X$. This implies that

$$d(J_o f_2, f_2) \leq \frac{K}{48^\beta}K^5. \tag{3.72}$$

Define a mapping $J_o : \Omega \to \Omega$ by

$$(J_o g)(x) = 2^{-1}g(2x),$$

for all $x \in X$ and for all $g \in \Omega$. We want to show that

$$d(J_o g, J_o h) \leq \frac{L}{2^\beta}d(g, h), \tag{3.73}$$

for all $g, h \in \Omega$. Let $g, h \in \Omega$. If $d(g, h) = \infty$ for all $g, h \in \Omega$, then the inequality (3.73) holds. So we may assume that $d(g, h) < \infty$. Assume that

$$C_o := \{c \in \mathbb{R}^+ | \|g(x) - h(x)\| \leq c\tilde{\psi}(x) \text{ for all } x \in X\}. \tag{3.74}$$

Since $d(g,h) < \infty$, we obtain $C_o \neq \emptyset$. Suppose that $c \in C_o$. For each $x \in X$, we have

$$\begin{aligned}\|(J_og)(x) - (J_o)h(x)\| &= \left\|2^{-1}g(2x) - 2^{-1}h(2x)\right\| \\ &= 2^{-\beta}\left\|g(2x) - h(2x)\right\| \\ &\leq \frac{c}{2^\beta}\tilde{\psi}(2x) \\ &\leq \frac{cL}{2^\beta}\tilde{\psi}(x),\end{aligned}$$

and so

$$d(J_og, J_oh) \leq \frac{cL}{2^\beta}.$$

By taking the infimum on $c \in C_o$, we obtain

$$d(J_og, J_oh) \leq \frac{L}{2^\beta}d(g,h).$$

Therefore, we can conclude that

$$d(J_og, J_oh) \leq \frac{L}{2^\beta}d(g,h).$$

for all $g, h \in \Omega$. By taking a function $\varphi : [0,\infty) \to [0,\infty)$ in Theorem 2.5 by

$$\varphi(t) = \frac{L}{2^\beta}t,$$

for all $t \in [0,\infty)$, there is the unique fixed point A of J_o in Ω such that $\{J_o^n f_2\}$ converges to A in (Ω, d). By Remark 2.7 and (3.72), we get

$$\begin{aligned}d(A, f_2) &\leq \left(\frac{K}{1 - 2^{-\beta}KL}\right)d(J_of_2, f_2) \\ &\leq \frac{K^2}{48^\beta - 24^\beta KL},\end{aligned} \tag{3.75}$$

for all $x \in X$. By (3.45), we have

$$\begin{aligned}\left\|\frac{Df_2(2^nx, 2^ny)}{2^n}\right\|^p &= \frac{1}{2^{n\beta p}}\left\|\frac{Df(2^nx, 2^ny) + Df(-2^nx, -2^ny)}{2}\right\|^p \\ &\leq \frac{1}{2^{\beta p + n\beta p}}\left(\|Df(2^nx, 2^ny)\|^p + \|Df(-2^nx, -2^ny)\|\right)^p \\ &\leq \frac{1}{2^{\beta p + n\beta p}}\left(\phi^p(2^nx, 2^ny) + \phi^p(-2^nx, -2^ny)\right) \\ &\leq \frac{1}{2^{\beta p + n\beta p}}L^{np}\left(\phi^p(x,y) + \phi^p(-x,-y)\right) \\ &= \frac{1}{2^{\beta p}}\left(\frac{L}{2^\beta}\right)^{np}\left(\phi^p(x,y) + \phi^p(-x,-y)\right),\end{aligned}$$

for all $x, y \in X$. Letting $n \to \infty$ in the last inequality, we have

$$DA(x,y) = 0, \tag{3.76}$$

for all $x, y \in X$. By using the fact that $f_2(0) = 0$, we obtain $A(0) = 0$. Since f_2 is odd, it yields that

$$\begin{aligned}\|A(x) + A(-x)\| &= \left\|A(x) + \lim_{n\to\infty} J_o^n f_2(-x)\right\| \\ &= \left\|A(x) - \lim_{n\to\infty} J_o^n f_2(x)\right\| \\ &= 0,\end{aligned}$$

for all $x \in X$ and so A is odd. By Lemma 3.2, we have A is an additive mapping. From (3.75) we get

$$\left\|\frac{f(x) - f(-x)}{2} - A(x)\right\| \leq \frac{K^2}{48^\beta - 24^\beta KL}\tilde{\psi}(x), \tag{3.77}$$

for all $x \in X$. Since $f(x) = f_1(x) + f_2(x)$ for all $x \in X$, from (3.68) and (3.77) it follows that

$$\begin{aligned}\|f(x) - Q(x) - A(x)\| &= \|f_1(x) + f_2(x) - Q(x) - A(x)\| \\ &\leq K\left(\|f_1(x) - Q(x)\| + |f_2(x) - A(x)\|\right) \\ &\leq K\left(\frac{11^\beta K^6}{480^\beta - 30^\beta KL} + \frac{K^2}{48^\beta - 24^\beta KL}\right)\tilde{\psi}(x) \qquad (3.78) \\ &= \left(\frac{11^\beta K^7}{480^\beta - 30^\beta KL} + \frac{K^3}{48^\beta - 24^\beta KL}\right)\tilde{\psi}(x), \qquad (3.79)\end{aligned}$$

for all $x \in X$.

Now, we show the uniqueness of Q and A. Suppose that $Q', A' : X \to Y$ satisfies (3.46) and (3.47), respectively. Since $Q(2x) = 16Q(x)$, by using (3.46), we have

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \left\|\frac{Q(2^n x)}{16^n} - \frac{Q'(2^n x)}{16^n}\right\| \\ &= \frac{1}{16^{n\beta}}\|Q(2^n x) - f_1(2^n x) - Q'(2^n x) + f_1(2^n x)\| \\ &\leq \frac{K}{16^{n\beta}}\left(\|Q(2^n x) - f_1(2^n x)\| + \|Q'(2^n x) - f_1(2^n x)\|\right) \\ &\leq \frac{2K}{16^{n\beta}}\left(\frac{11^\beta K^6}{480^\beta - 30^\beta KL}\right)\tilde{\psi}(2^n x) \\ &\leq \frac{2K}{16^{n\beta}}\left(\frac{11^\beta K^6}{480^\beta - 30^\beta KL}\right)L^n\tilde{\psi}(x),\end{aligned}$$

for all $x \in X$. Since the right-hand side of the above inequality converges to 0 as $n \to \infty$, we obtain that $Q(x) = Q'(x)$ for all $x \in X$ and so $Q = Q'$. Similarly, we get $A = A'$. ∎

Next, we give the stability result which is similar to Theorem 3.3. In order to avoid repetition, the proof of this result is omitted.

Theorem 3.4. *Let X be a normed space, Y be a $(\beta, p)-$Banach space with the modulus of concavity K and $\phi : X \times X \to [0, \infty)$ be a function such that*

$$\phi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2^{4\beta}}\phi(x, y), \tag{3.80}$$

for all $x, y \in X$*, where* $0 \leq L < 1$ *with* $KL < 1$*. Suppose that* $f : X \to Y$ *is a mapping satisfying* $f(0) = 0$ *and*

$$\|Df(x,y)\| \leq \phi(x,y), \tag{3.81}$$

for all $x, y \in X$*. Then there exist a unique quartic function* $Q : X \to Y$ *and a unique additive function* $A : X \to Y$ *such that*

$$\left\|\frac{f(x)+f(-x)}{2} - Q(x)\right\| \leq \frac{11^\beta K^6}{30^\beta - 30^\beta KL}\tilde{\psi}(x), \tag{3.82}$$

$$\left\|\frac{f(x)-f(-x)}{2} - A(x)\right\| \leq \frac{K^2}{24^\beta - 3^\beta KL}\tilde{\psi}(x) \tag{3.83}$$

and

$$\|f(x) - Q(x) - A(x)\| \leq \left(\frac{11^\beta K^7}{30^\beta - 30^\beta KL} + \frac{K^3}{24^\beta - 3^\beta KL}\right)\tilde{\psi}(x), \tag{3.84}$$

for all $x \in X$*, where*

$$\tilde{\psi}(x) := \psi\left(\frac{3x}{2}, \frac{x}{2}\right) + \psi\left(x, \frac{x}{2}\right) + \psi\left(\frac{x}{2}, \frac{x}{2}\right) + \psi\left(0, \frac{x}{2}\right) + \psi(0,x)$$

such that $\psi(x,y) := \phi(x,y) + \phi(-x,-y)$*.*

4. Conclusions

The main results of this paper are two stability results for the additive-quartic functional equation (1.7). These results can be applied to many stability results by taking the specific control function ϕ. For instance, the readers can take the function ϕ in Theorem 3.3 by

$$\phi(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0; \\ \lambda\left(\|x\|^s + \|y\|^s\right), & \text{otherwise} \end{cases}$$

or

$$\phi(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0; \\ \lambda\left(\|x\|^s \|y\|^s + \|x\|^s + \|y\|^s\right), & \text{otherwise,} \end{cases}$$

where λ is a positive real number and s is a negative real number such that $2^s K < 1$. Furthermore, the readers can use the technique in the proof in this paper for investigating the stability results for various kinds of functional equations in quasi-β-Banach spaces.

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