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A Generalization of Subnexuses Based on $\mathcal{N}$-Structures

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Abstract In this paper, we generalize the concepts of $\mathcal{N}$-subnexuses of types $(\in, q)$, $(\in, \in \vee q)$ and $(q, \in \vee q)$, and introduce the notions of $\mathcal{N}$-subnexuses of types $(\in, q_k)$, $(\in, \in \vee q_k)$ and $(q, \in \vee q_k)$. We investigate their basic properties, characterize subnexuses by $\mathcal{N}$-subnexuses of type $(\in, \in \vee q_k)$, and give some characterizations for $\mathcal{N}$-subnexuses of types $(\in, q_k)$ and $(q, \in \vee q_k)$. Moreover, we define $\mathcal{N}$-subnexuses of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ and discuss on their different properties.

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1. Introduction

Nexuses are a type of structure algebras which defined by M. Bolourian in [1], where some properties of them such as sub-nexuses, cyclic nexuses and homomorphism of nexuses were investigated. Next, studies from algebraic view generalized on nexuses. D. Afkhami et al. [2] defined the notion of fraction over a nexus and studied its basic properties. Moreover D. Afkhami et al. [3] defined the soft nexuses over a nexus and studied the prime and maximal soft subnexuses over a nexus. H. Hedayati et al. [4] introduced normal, maximal and product fuzzy subnexuses of a nexus. Also, about applications of nexuses can see [5] and [6].

After appearance of (α, β)-fuzzy substructures, based on the concepts of belongingness and quasi-coincidence for a fuzzy point of a fuzzy subset, those defined and studied on many algebraic structures which some of them can be seen in [7–14]. On the other hand, Jun et al. [15] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures, as a mathematical tool for dealing with negative information

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(beside, fuzzy sets which relied on spreading positive information). They discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in $BCK/BCI/BCH$-algebras (see [15–19]).

By combining the above concepts, Norouzi et al. [20] introduced the notion of a subnexus based on $\mathcal{N}$-function (briefly, $\mathcal{N}$-subnexus), and investigated related properties. They discussed characterization of $\mathcal{N}$-subnexus. They also introduced the notion of $\mathcal{N}$-subnexus of type (α, β) with

$$(\alpha, \beta) \in \{(\in, \in), (\in, q), (\in, \in \vee q), (q, \in), (q, q), (q, \in \vee q)\},$$

and investigated their basic properties. Now, in this paper, we generalize the concepts in [20] and introduce the notion of $\mathcal{N}$-subnexus of type $(\in, q_k)$, $(\in, \in \vee q_k)$, (q, q_k), $(q, \in \vee q_k)$, and also investigate basic properties of them. In this way, connection of the notions is studied. Characterizations of $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ are given. Conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subnexus of type $(q, \in \vee q_k)$ are provided. Moreover, the notion of $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ is defined and some characterizations of it are established, where we can see some differences with other similar (α, β)-substructures.

2. Preliminaries

In this section we give some definitions and results which we need to develop our paper. They have been brought of [3, 4, 21], in connection with nexuses, and [18, 19] in connection with $\mathcal{N}$-structures.

An *address* is a sequence of $N^* = \mathbb{N} \cup \{0\}$ such that $a_k = 0$ implies that $a_i = 0$ for all $i \geq k$. The sequence of zero is called the *empty address* and denoted by ( ). In other word, every nonempty address is of the form $(a_1, a_2, \cdots, a_n, 0, 0, \cdots)$ where $n \in \mathbb{N}$, and it is denoted by $(a_1, a_2, \cdots, a_n)$.

Definition 2.1. A set X of addresses is called a *nexus* if

(1) $(a_1, a_2, \ldots, a_n) \in X$ implies that $(a_1, \ldots, a_{n-1}, t) \in X$ for all $0 \leq t \leq a_n$.
(2) $(a_i)_{i=1}^{\infty} \in X$ implies that $(a_1, a_2, \ldots, a_n) \in X$ for all $n \in \mathbb{N}$.

Example 2.2. A set $X = \{(), (1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2)\}$ is a nexus. But, $X' = \{(), (1), (2), (2, 2)\}$ is not a nexus since $(2, 2)$ is an element of X' but $(2, 1) \notin X'$.

Let X be a nexus and $w \in X$. The *level* of w, denoted by $l(w)$, is said to be:

(i) 0 if $w = ()$.
(ii) n if $w = (a_1, a_2, \ldots, a_n)$ for some $a_n \in \mathbb{N}$.
(iii) ∞ if w is an infinite sequence of $\mathbb{N}$.

Definition 2.3. Let $v = (a_i)$ and $w = (b_i)$ be addresses where $a_i, b_i \in \mathbb{N}$. Then $v \leq w$ if $l(v) = 0$ or one of the following cases is satisfied:

(i) If $l(v) = 1$, i.e., $v = (a_1)$ for $a_1 \in \mathbb{N}$, then $l(w) \geq 1$ and $a_1 \leq b_1$.
(ii) If $1 < l(v) < \infty$, then $l(v) \leq l(w)$ and $a_{l(v)} \leq b_{l(v)}$ and for every $1 \leq i < l(v)$ we have, $a_i = b_i$.
(iii) If $l(w) = \infty$, then $v = w$.

Definition 2.4. A nonempty subset S of a nexus X is called a *subnexus* of X if S itself is a nexus. The set of all subnexuses of X is denoted by $SUB(X)$.

Note that a subset S of a nexus X is a subnexus of X if and only if it satisfies:

$$(\forall v, w \in X)(v \leq w,\ w \in S \ \Rightarrow\ v \in S). \tag{2.1}$$

Example 2.5. Consider a nexus

$$X = \{(), (1), (2), (3), (1,1), (2,1), (3,1), (3,1,1), (3,1,2)\}.$$

Then $X_1 = \{(), (1), (2), (3), (2,1)\}$, $X_2 = \{(), (1), (2), (1,1), (2,1)\}$ and $X_3 = \{(), (1), (2), (3), (3,1)\}$ are subnexuses of X.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Let $F(X, [-1,0])$ be the set of all functions from the set X to $[-1,0]$ (for briefly every element of $F(X, [-1,0])$ is said to be $\mathcal{N}$-function on X). An $\mathcal{N}$-structure is a pair (X, f) of X and an $\mathcal{N}$-function f on X. For any $\mathcal{N}$-structure (X, f) and $\alpha \in [-1, 0)$, the set $C(f;\alpha) = \{x \in X \mid f(x) \leq \alpha\}$ is called the *closed support* of (X, f) related to α, and the set $O(f;\alpha) = \{x \in X \mid f(x) < \alpha\}$ is said to be the *open support* of (X, f) related to α.

Let $\alpha \in [-1, 0)$ and (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f(y) = \begin{cases} 0 & \text{if } y \neq x, \\ \alpha & \text{if } y = x. \end{cases}$$

In this case, f is denoted by x_α, and (X, x_α) is said to be a *point $\mathcal{N}$-structure with support x and value α*. For any $\mathcal{N}$-structure (X, g), we say that a point $\mathcal{N}$-structure (X, x_α) is an *$\mathcal{N}_\in$-subset* (resp. *$\mathcal{N}_q$-subset*) of (X, g) if $g(x) \leq \alpha$ (resp. $g(x) + \alpha + 1 < 0$). If a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_\in$-subset or an $\mathcal{N}_q$-subset of (X, g), then we say (X, x_α) is an *$\mathcal{N}_{\in\vee q}$-subset* of (X, g).

3. A Generalization of Subnexuses by $\mathcal{N}$-Function

In what follows, let X and (α, k) be a nexus and an arbitrary element of $[-1, 0) \times (-1, 0]$, respectively, unless otherwise specified.

For any $\mathcal{N}$-structure (X, g), we say that a point $\mathcal{N}$-structure (X, x_α) is an *$\mathcal{N}_{q_k}$-subset* of (X, g) if $g(x) + \alpha - k + 1 < 0$). If a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_\in$-subset or an $\mathcal{N}_{q_k}$-subset of (X, g), then we say (X, x_α) is an *$\mathcal{N}_{\in\vee q_k}$-subset* of (X, g).

Definition 3.1 ([20])**.** By a *subnexus* of X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-subnexus* of X), we mean an $\mathcal{N}$-structure (X, f) in which f satisfies the following assertion:

$$(\forall v, w \in X)\,(w \leq v \;\Rightarrow\; f(w) \leq f(v))\,. \tag{3.1}$$

Definition 3.2 ([20])**.** An $\mathcal{N}$-subnexus (X, f) is said to be of type

(i) $(\in, \in)$ (resp., $(\in, q)$ and $(\in, \in\vee q)$) if whenever the point $\mathcal{N}$-structure (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f) then the point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_\in$-subset (resp., $\mathcal{N}_q$-subset and $\mathcal{N}_{\in\vee q}$-subset) of (X, f) for all $v, w \in X$ with $v \leq w$.

(ii) $(q, \in)$ (resp., (q, q) and $(q, \in\vee q)$) if whenever the point $\mathcal{N}$-structure (X, w_α) is an $\mathcal{N}_q$-subset of (X, f) then the point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_\in$-subset (resp., $\mathcal{N}_q$-subset and $\mathcal{N}_{\in\vee q}$-subset) of (X, f) for all $v, w \in X$ with $v \leq w$.

Definition 3.3. An $\mathcal{N}$-subnexus (X, f) is said to be of type

- $(\in, q_k)$ if whenever the point $\mathcal{N}$-structure (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f) then the point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_{q_k}$-subset of (X, f) for all $v, w \in X$ with $v \leq w$.
- $(\in, \in \vee q_k)$ if whenever the point $\mathcal{N}$-structure (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f) then the point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f) for all $v, w \in X$ with $v \leq w$.
- $(q, \in \vee q_k)$ if whenever the point $\mathcal{N}$-structure (X, w_α) is an $\mathcal{N}_q$-subset of (X, f) then the point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f) for all $v, w \in X$ with $v \leq w$.

Example 3.4. Let (X, f) be an $\mathcal{N}$-structure in which

$$X = \{(), (1), (2), (1,1), (1,2), (1,3), (1,3,1), (1,3,2)\}$$

is a nexus and f is defined as follows:

$$f = \begin{pmatrix} () & (1) & (2) & (1,1) & (1,2) & (1,3) & (1,3,1) & (1,3,2) \\ -1 & -0.9 & -0.93 & -0.95 & -0.94 & -0.96 & -0.97 & -0.99 \end{pmatrix}.$$

Put $k = -0.75$. It is easy to see that in the nexus X we have

$$\begin{aligned} (1) &\leq (2),\ (1,1),\ (1,2),\ (1,3),\ (1,3,1),\ (1,3,2) \\ (1,1) &\leq (1,2),\ (1,3),\ (1,3,1),\ (1,3,2) \\ (1,2) &\leq (1,3),\ (1,3,1),\ (1,3,2) \\ (1,3) &\leq (1,3,1),\ (1,3,2) \\ (1,3,1) &\leq (1,3,2). \end{aligned}$$

Since, $() \leq v$ and $f(()) \leq f(v)$ for all $v \in X$, clearly if (X, v_α) is an $\mathcal{N}_\in$-subset of (X, f) then $(X, ()_\alpha)$ is an $\mathcal{N}_\in$-subset of (X, f) for all $\alpha \in [-1, 0)$. For $(1) \leq (2)$ we have $f(2) = -0.93 < \beta$ and $f(1) = -0.9 \not< \beta$ for all $\beta \in (-0.93, -0.9)$, but $f(1) + \beta + 0.75 + 1 < 0$. This means that if $(X, (2)_\beta)$ is an $\mathcal{N}_\in$-subset of (X, f) then $(X, (1)_\beta)$ is an $\mathcal{N}_{q_{-0.75}}$-subset of (X, f). For $(1,1) \leq (1,2)$, since $f(1,1) \leq f(1,2)$, if $(X, (1,2)_\alpha)$ is an $\mathcal{N}_\in$-subset of (X, f) then $(X, (1,1)_\alpha)$ is an $\mathcal{N}_\in$-subset of (X, f) for all $\alpha \in (-0.94, 0)$. For $(1,1) \leq (1,3)$ and $\beta \in (-0.96, -0.95)$, if $(X, (1,3)_\beta)$ is an $\mathcal{N}_\in$-subset of (X, f) then $(X, (1,1)_\beta)$ is an $\mathcal{N}_{q_{-0.75}}$-subset of (X, f) while is not an $\mathcal{N}_\in$-subset of (X, f). By a similar manner, we can see the related implication is valid for all other cases. Therefore, (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ with $k = -0.75$.

Example 3.5. Consider the nexus $X = \{(), (1), (1,1), (1,2), (1,3)\}$ with an $\mathcal{N}$-function f is defined as follows:

$$f = \begin{pmatrix} () & (1) & (1,1) & (1,2) & (1,3) \\ -0.73 & -0.74 & -0.75 & -0.76 & -0.8 \end{pmatrix}.$$

It is easy to see that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, q_k)$ with $k = -0.4$.

Example 3.6. Define an $\mathcal{N}$-function g on the set $X = \{(), (1), (2), (2,1), (2,2)\}$ as:

$$g = \begin{pmatrix} () & (1) & (2) & (2,1) & (2,2) \\ -0.9 & -0.8 & -0.7 & -0.5 & -0.3 \end{pmatrix}.$$

Then (X, g) is an $\mathcal{N}$-subnexus of type $(q, \in \vee q_k)$ with $k = -0.1$.

We note that every $\mathcal{N}$-subnexus of type $(\in, q_k)$ (resp., $(\in, \in \vee q_k)$ and $(q, \in \vee q_k)$) with $k = 0$ is an $\mathcal{N}$-subnexus of type $(\in, q)$ (resp., $(\in, \in \vee q)$ and $(q, \in \vee q)$). But the converse is not true in general as seen in the following example.

Example 3.7. (1) Consider the nexus $X = \{(), (1), (2), (1,1), (1,2)\}$ and the $\mathcal{N}$-function f on X defined as

$$f = \begin{pmatrix} () & (1) & (2) & (1,1) & (1,2) \\ -1 & -0.7 & -0.73 & -0.74 & -0.75 \end{pmatrix}.$$

It can be seen that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ which is not of type $(\in, \in \vee q_k)$ for $k = -0.75$. Indeed, we have $(1) \leq (2)$ and $(X, (2)_{-0.73})$ is an $\mathcal{N}_\in$-subset of (X, f), but $f((1)) \not\leq -0.73$ and $f((1)) - 0.73 - k + 1 = 0.32 \not< 0$. This implies that $(X, (1)_{-0.73})$ is not an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f) and so (X, f) is not an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ for $k = -0.75$.

(2) Let (X, f) be an $\mathcal{N}$-structure in which $X = \{(), (1), (1,1), (1,2)\}$ is a nexus and g is defined as follows:

$$g = \begin{pmatrix} () & (1) & (1,1) & (1,2) \\ -0.64 & -0.62 & -0.63 & -0.71 \end{pmatrix}.$$

Then (X, g) is an $\mathcal{N}$-subnexus of type $(\in, q)$, but (X, g) is not an $\mathcal{N}$-subnexus of type $(\in, q_k)$ for $k = -0.4$, since $(1,1) \leq (1,2)$, $g((1,2)) \leq -0.71$, $g((1,1)) \not\leq -0.71$ and $g((1,1)) - 0.71 + 0.4 + 1 \not< 0$.

(3) An $\mathcal{N}$-structure (X, f) in which $X = \{(), (1), (2), (1,2)\}$ is a nexus and f is defined as follows:

$$f = \begin{pmatrix} () & (1) & (2) & (1,2) \\ -0.9 & -0.91 & -0.92 & -0.93 \end{pmatrix}.$$

is an $\mathcal{N}$-subnexus of type $(q, \in \vee q)$, but is not an $\mathcal{N}$-subnexus of type $(q, \in \vee q_k)$ for $k = -0.9$.

In the following, we give some characterizations for an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$.

Theorem 3.8. *An $\mathcal{N}$-subnexus (X, f) is of type $(\in, \in \vee q_k)$ if and only if the following assertion is valid.*

$$(\forall v, w \in X) \left(v \leq w \;\Rightarrow\; f(v) \leq \bigvee \left\{ f(w), \frac{k-1}{2} \right\} \right). \tag{3.2}$$

Proof. Suppose that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$. For any $v, w \in X$, assume that $v \leq w$ and $f(w) > \frac{k-1}{2}$. If $f(v) > f(w)$, then there exists $\beta \in [-1, 0)$ such that $f(v) > \beta \geq f(w)$. Thus the point $\mathcal{N}$-structure (X, w_β) is an $\mathcal{N}_\in$-subset of (X, f), but the point $\mathcal{N}$-structure (X, v_β) is not an $\mathcal{N}_\in$-subset of (X, f). Also

$$f(v) + \beta - k + 1 > 2\beta - k + 1 \geq 2f(w) - k + 1 > 0,$$

and so (X, v_β) is not an $\mathcal{N}_{q_k}$-subset of (X, f). Therefore (X, v_β) is not an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f), which is a contradiction. Hence $f(v) \leq f(w)$ whenever $f(w) > \frac{k-1}{2}$. Now, suppose that $f(w) \leq \frac{k-1}{2}$. Then the point $\mathcal{N}$-structure $\left(X, w_{\frac{k-1}{2}}\right)$ is an $\mathcal{N}_\in$-subset of (X, f) and so $\left(X, v_{\frac{k-1}{2}}\right)$ is an $\mathcal{N}_{\in \vee q}$-subset of (X, f) by hypothesis. If $\left(X, v_{\frac{k-1}{2}}\right)$ is an $\mathcal{N}_\in$-subset of (X, f) then $f(v) \leq \frac{k-1}{2}$ and so $f(v) \leq \bigvee\{f(w), \frac{k-1}{2}\}$. If $\left(X, v_{\frac{k-1}{2}}\right)$ is an

$\mathcal{N}_{q_k}$-subset of (X, f), then $f(v) + \frac{k-1}{2} - k + 1 < 0$, that is, $f(v) < \frac{k-1}{2}$. Consequently $f(v) \leq \bigvee\{f(w), \frac{k-1}{2}\}$.

Conversely, assume that (3.2) is valid. Let $v, w \in X$ and $\beta \in [-1, 0)$ be such that $v \leq w$ and the point $\mathcal{N}$-structure (X, w_β) is an $\mathcal{N}_\in$-subset of (X, f). If $f(v) \leq \beta$, then the point $\mathcal{N}$-structure (X, v_β) is an $\mathcal{N}_\in$-subset of (X, f). Suppose that $f(v) > \beta$. Then $f(w) \leq \beta < f(v) \leq \bigvee \{f(w), \frac{k-1}{2}\}$, and therefore $\bigvee\{f(w), \frac{k-1}{2}\} = \frac{k-1}{2}$. It follows that

$$f(v) + \beta - k + 1 < 2f(v) - k + 1 \leq 2\left(\bigvee\{f(w), \frac{k-1}{2}\}\right) - k + 1 = 0.$$

Thus (X, v_β) is an $\mathcal{N}_{q_k}$-subset of (X, f). Consequently (X, v_β) is an $\mathcal{N}_{\in\vee q_k}$-subset of (X, f) and thus (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$. ∎

Corollary 3.9 ([20]). *An $\mathcal{N}$-subnexus (X, f) is of type $(\in, \in \vee q)$ if and only if the following assertion is valid.*

$$(\forall v, w \in X)\left(v \leq w \ \Rightarrow \ f(v) \leq \bigvee \{f(w), -0.5\}\right).$$

Proposition 3.10. *If (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$, then*

$$(\forall v \in X)\left(f(()) \leq \bigvee\{f(v), \frac{k-1}{2}\}\right).$$

Proof. Since $() \leq v$ for all $v \in X$, it is straightforward. ∎

Corollary 3.11. *If (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$, then*

$$(\forall v \in X)(f(()) \leq \bigvee\{f(v), -0.5\}.$$

Theorem 3.12. *Let (X, f) be an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$. Then*

(1) *if there exists $x \in X$ such that $f(x) \leq \frac{k-1}{2}$, then $f(()) \leq \frac{k-1}{2}$.*
(2) *if $f(()) > \frac{k-1}{2}$, then (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$.*

Proof. (1) Assume that there exists $x \in X$ such that $f(x) \leq \frac{k-1}{2}$. If $x = ()$, it is true. If $x \neq ()$, then $f(()) \leq \bigvee\{f(x), \frac{k-1}{2}\} = \frac{k-1}{2}$ by Theorem 3.8 and hypothesis.

(2) Suppose that $f(()) > \frac{k-1}{2}$ and $f(v) > f(w)$ for all $v, w \in X$ with $v \leq w$. It follows from Theorem 3.8 that $f(v) \leq \bigvee\{f(w), \frac{k-1}{2}\} = \frac{k-1}{2}$. Since $() \leq w$, we have $f(()) \leq \bigvee\{f(w), \frac{k-1}{2}\} = \frac{k-1}{2}$. This is a contradiction, and hence $f(v) \leq f(w)$ for all $v, w \in X$ with $v \leq w$. Therefore (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$. ∎

Corollary 3.13 ([20]). *Let (X, f) be an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$. Then*

(1) *if there exists $x \in X$ such that $f(x) \leq -0.5$, then $f(()) \leq -0.5$.*
(2) *if $f(()) + 0.5 > 0$, then (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$.*

Theorem 3.14. *If $-1 < k < r \leq 0$, then every $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_r)$.*

Proof. Let (X, f) be an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$. Then

$$f(v) \leq \bigvee \left\{ f(w), \frac{k-1}{2} \right\} \leq \bigvee \left\{ f(w), \frac{r-1}{2} \right\},$$

for all $v, w \in X$ with $v \leq w$. It follows from Theorem 3.8 that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee qr)$. ■

The following example shows that if $-1 < k < r \leq 0$, then an $\mathcal{N}$-subnexus of type $(\in, \in \vee qr)$ may not be an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$.

Example 3.15. The $\mathcal{N}$-structure (X, f) defined in Example 3.4 is an $\mathcal{N}$-subnexus of type $(\in, \in \vee qr)$ for $r = -0.75$, but it is not an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ for $k = -0.9$. Indeed, $(1) \leq (2)$, but $f((1)) = -0.9 \not\leq -0.93 = \bigvee\{-0.93, -0.95\} = \bigvee\{f((2)), \frac{k-1}{2}\}$.

Theorem 3.16. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ if and only if for every $\alpha \in [\frac{k-1}{2}, 0]$ the nonempty closed support of (X, f) related to α is a subnexus of X.*

Proof. Assume that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ and let $\alpha \in [\frac{k-1}{2}, 0]$ such that $C(f; \alpha) \neq \emptyset$. Let $v \leq w$ and $w \in C(f; \alpha)$. Then $f(v) \leq \bigvee\{f(w), \frac{k-1}{2}\}$ by Theorem 3.8. If $\bigvee\{f(w), \frac{k-1}{2}\} = f(w)$, then $f(v) \leq f(w) \leq \alpha$ and thus $v \in C(f; \alpha)$. Also, if $\bigvee\{f(w), \frac{k-1}{2}\} = \frac{k-1}{2}$, then $f(v) \leq \frac{k-1}{2} \leq \alpha$, and thus $v \in C(f; \alpha)$. Hence $C(f; \alpha)$ is a subnexus of X.

Conversely, let (X, f) be an $\mathcal{N}$-structure such that the nonempty closed support of (X, f) related to α is a subnexus of X for all $\alpha \in [\frac{k-1}{2}, 0]$. If there exist $v, w \in X$ such that $v \leq w$ and $f(v) > \bigvee\{f(w), \frac{k-1}{2}\}$, then we can take $\beta \in [-1, 0]$ such that $f(v) > \beta \geq \bigvee\{f(w), \frac{k-1}{2}\}$. Thus $w \in C(f; \beta)$ and $\beta \geq \frac{k-1}{2}$. Since $C(f; \beta)$ is a subnexus of X, we have $v \in C(f; \beta)$. Hence $f(v) \leq \beta$, a contradiction. Therefore $f(v) \leq \bigvee\{f(w), \frac{k-1}{2}\}$ for all $v, w \in X$. It follows from Theorem 3.8 that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$. ■

Corollary 3.17 ([20])**.** *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ if and only if for every $\alpha \in [-0.5, 0]$ the nonempty closed support of (X, f) related to α is a subnexus of X.*

Theorem 3.18. *Let S be a subnexus of X. For any $\alpha \in (\frac{k-1}{2}, 0)$, there exists an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ for which S is represented by the closed support of (X, f) related to α.*

Proof. Let (X, f) be an $\mathcal{N}$- structure in which f is given by

$$f(x) = \begin{cases} \alpha & \text{if } x \in S, \\ 0 & \text{if } x \notin S, \end{cases}$$

for all $x \in X$ where $\alpha \in (\frac{k-1}{2}, 0)$. Assume that $f(\nu) > \bigvee\{f(\omega), \frac{k-1}{2}\}$ for some $\nu, \omega \in X$ such that $\nu \leq \omega$. By $|Im(f)| = 2$, it follows that $f(\nu) = 0$ and $\bigvee\{f(\omega), \frac{k-1}{2}\} = \alpha$. Since $\alpha > \frac{k-1}{2}$, we have $f(\omega) = \alpha$ and so $\omega \in S$. Since S is a subnexus of X, we obtain $\nu \in S$ and thus $f(\nu) = \alpha < 0$, which is a contradiction. Therefore $f(\nu) \leq \bigvee\{f(\omega), \frac{k-1}{2}\}$ for all $\nu, \omega \in X$. Hence, (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$, by Theorem 3.8. Obviously, S is represented by the closed support of (X, f) related to α. ■

Corollary 3.19 ([20]). *Let S be a subnexus of X. For any $\alpha \in (-0.5, 0)$, there exists an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ for which S is represented by the closed support of (X, f) related to α.*

Note that every $\mathcal{N}$-subnexus of type $(\in, \in)$ is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$. But the converse is not true in general as seen in the following example.

Example 3.20. The $\mathcal{N}$-structure (X, f) defined in Example 3.4 is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$, but is not an $\mathcal{N}$-subnexus of type $(\in, \in)$, since $(1) \leq (2)$ but $f((1)) = -0.9 \nleq -0.93 = f((2))$.

Now, we give a condition for an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ to be an $\mathcal{N}$-subnexus of type $(\in, \in)$.

Theorem 3.21. *Let (X, f) be an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ such that $f(x) \geq \frac{k-1}{2}$ for all $x \in X$. Then (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$.*

Proof. Let $v, w \in X$ such that $v \leq w$ and (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f) for $\alpha \in [-1, 0)$. Then $f(w) \leq \alpha$. It follows from Theorem 3.8 and the hypothesis that

$$f(v) \leq \bigvee\{f(w), \frac{k-1}{2}\} = f(w) \leq \alpha.$$

Thus (X, v_α) is an $\mathcal{N}_\in$-subset of (X, f).
Therefore (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$. ■

For any $\mathcal{N}$-structure (X, f) and $\alpha \in [-1, 0)$, the *q_k-support* and the *$\in \vee q_k$-support* of (X, f) related to α are defined as follow

$$\mathcal{N}_{q_k}(f; \alpha) = \{x \in X \mid (X, x_\alpha) \text{ is an } \mathcal{N}_{q_k}\text{-subset of } (X, f)\}, \text{ and}$$
$$\mathcal{N}_{\in \vee q_k}(f; \alpha) = \{x \in X \mid (X, x_\alpha) \text{ is an } \mathcal{N}_{\in \vee q_k}\text{-subset of } (X, f)\}.$$

Note that the $\in \vee q_k$-support is the union of the closed support and the q_k-support, that is, $\mathcal{N}_{\in \vee q_k}(f; \alpha) = C(f; \alpha) \cup \mathcal{N}_{q_k}(f; \alpha)$.

Theorem 3.22. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$ if and only if the $\in \vee q_k$-support of (X, f) related to α is a subnexus of X for all $\alpha \in [-1, 0)$.*

Proof. Suppose that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q_k)$. Let $v, w \in X$ such that $v \leq w$ and $w \in \mathcal{N}_{\in \vee q_k}(f; \alpha)$ for $\alpha \in [-1, 0)$, $k \in (-1, 0]$. So (X, w_α) is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f). Thus $f(w) \leq \alpha$ or $f(w) + \alpha - k + 1 < 0$. If $f(w) \leq \alpha$, then (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f). By hypothesis (X, v_α) is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f) and so $v \in \mathcal{N}_{\in \vee q}(f; \alpha)$. If $f(w) + \alpha - k + 1 < 0$, we consider the following two cases:
If $\alpha \geq \frac{k-1}{2}$, then by hypothesis and Theorem 3.8, $f(v) \leq \bigvee\{f(w), \frac{k-1}{2}\} = \frac{k-1}{2} \leq \alpha$. So (X, v_α) is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f) and so $v \in \mathcal{N}_{\in \vee q_k}(f; \alpha)$. If $\alpha < \frac{k-1}{2}$, then we have:

i) If $\bigvee\{f(w), \frac{k-1}{2}\} = \frac{k-1}{2}$, then by hypothesis and Theorem 3.8,

$$f(v) + \alpha - k + 1 < f(v) + \frac{1-k}{2} \leq \bigvee\{f(w), \frac{k-1}{2}\} + \frac{1-k}{2} = 0.$$

ii) If $\bigvee\{f(w), \frac{k-1}{2}\} = f(w)$, then

$$f(v) + \alpha - k + 1 \leq \bigvee\{f(w), \frac{k-1}{2}\} + \alpha - k + 1 = f(w) + \alpha - k + 1 < 0.$$

Thus in each case we have $f(v) + \alpha - k + 1 < 0$ and so (X, v_α) is an $\mathcal{N}_{q_k}$-subset of (X, f). Consequently $v \in \mathcal{N}_{\in\vee q_k}(f;\alpha)$. Therefore $\mathcal{N}_{\in\vee q_k}(f;\alpha)$ is a subnexus of X for all $\alpha \in [-1, 0),\ k \in (-1, 0]$.
Conversely, let (X, f) be an $\mathcal{N}$-structure for which $(\in\vee q_k)$-support of (X, f) related to α is a subnexus of X for all $\alpha \in [-1, 0)$ and $k \in (-1, 0]$. Assume that there exists $v, w \in X$ such that $v \le w$ and $f(v) > \bigvee\{f(w), \frac{k-1}{2}\}$. Then $f(v) > \beta \ge \bigvee\{f(w), \frac{k-1}{2}\}$ for some $\beta \in [\frac{k-1}{2}, 0)$. It follows that $w \in C(f;\beta) \subseteq \mathcal{N}_{\in\vee q_k}(f;\beta)$ but $v \notin C(f;\beta)$. Also $f(v) + \beta - k + 1 > 2\beta - k + 1 \ge 0$, that is $v \notin \mathcal{N}_{q_k}(f;\beta)$. Thus $v \notin \mathcal{N}_{\in\vee q_k}(f;\beta)$ which is a contradiction. Therefore $f(v) \le \bigvee\{f(w), \frac{k-1}{2}\}$ for all $v, w \in X$. Using Theorem 3.8, then (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in\vee q_k)$. ∎

Theorem 3.23. *If (X, f) is an $\mathcal{N}$-subnexus of type $(\in, q_k)$, then the set*

$$O(f;k) := \{x \in X \mid f(x) < k\}$$

is a subnexus of X.

Proof. Let (X, f) be an $\mathcal{N}$-subnexus of type $(\in, q_k)$ and $v, w \in X$ such that $v \le w$ and $w \in O(f;k)$. Note that $(X, w_{f(w)})$ is an $\mathcal{N}_\in$-subset of (X, f). If $f(v) \ge k$, then $f(v) + f(w) - k + 1 \ge k + f(w) - k + 1 \ge f(w) + 1 \ge 0$. Thus $(X, v_{f(w)})$ is not an $\mathcal{N}_{q_k}$-subset of (X, f), a contradiction. Hence $f(v) < k$, that is, $v \in O(f;k)$. Hence $O(f;k)$ is a subnexus of X. ∎

Corollary 3.24 ([20]). *If (X, f) is an $\mathcal{N}$-subnexus of type $(\in, q)$, then the open support of (X, f) relative 0 is a subnexus of X.*

Now, we provide conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subnexus of type $(q, \in\vee q_k)$.

Theorem 3.25. *Let S be a subnexus of X and let (X, f) be an $\mathcal{N}$-structure such that*

(1) $(\forall x \in X)(x \in S \ \Rightarrow\ f(x) \le \frac{k-1}{2})$,
(2) $(\forall x \in X)(x \notin S \ \Rightarrow\ f(x) = 0)$.

Then (X, f) is an $\mathcal{N}$-subnexus of type $(q, \in\vee q_k)$.

Proof. Let $v, w \in X$ with $v \le w$ and $(\alpha, k) \in [-1, 0) \times (-1, 0]$ be such that the point $\mathcal{N}$-structure (X, w_α) is an $\mathcal{N}_q$-subset of (X, f). Then $f(w) + \alpha + 1 < 0$. It implies that $v \in S$ since if $v \notin S$, then $w \notin S$. Thus $f(w) = 0$ and so $\alpha + 1 = f(w) + \alpha + 1 < 0$, that is, $\alpha < -1$, this is a contradiction. Therefore $f(v) \le \frac{k-1}{2}$. If $\alpha < \frac{k-1}{2}$, then $f(v) + \alpha - k + 1 < \frac{k-1}{2} + \frac{k-1}{2} - k + 1 = 0$ and thus the point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_{q_k}$-subset of (X, f). If $\alpha \ge \frac{k-1}{2}$, then $f(v) \le \frac{k-1}{2} \le \alpha$ and so the point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_\in$-subset of (X, f). Thus the point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_{\in\vee q_k}$-subset of (X, f), and therefore (X, f) is an $\mathcal{N}$-subnexus of type $(q, \in\vee q_k)$. ∎

Corollary 3.26 ([20]). *Let S be a subnexus of X and let (X, f) be an $\mathcal{N}$-structure such that*

(1) $(\forall x \in X)(x \in S \ \Rightarrow\ f(x) \le -0.5)$,
(2) $(\forall x \in X)(x \notin S \ \Rightarrow\ f(x) = 0)$.

Then (X, f) is an $\mathcal{N}$-subnexus of type $(q, \in\vee q)$.

Theorem 3.27. *Let (X, f) be an $\mathcal{N}$-subnexus of type $(q, \in \vee q_k)$. If f is not constant on $O(f;k)$ and $f(()) \geq f(x)$ for all $x \in X$, then there exists $y \in X$ such that $f(y) \leq \frac{k-1}{2}$. In particular, $f(()) \leq \frac{k-1}{2}$.*

Proof. Assume that $f(x) > \frac{k-1}{2}$ for all $x \in X$. Since f is not constant on $O(f;k)$, there exists $y \in O(f;k)$ such that $\alpha_y = f(y) \neq f(()) = \alpha_0$. Then $\alpha_0 > \alpha_y$. Choose $\beta < \frac{k-1}{2}$ such that $\alpha_0 + \beta - k + 1 > 0 > \alpha_y + \beta + 1$. Then the point $\mathcal{N}$-structure (X, y_β) is an $\mathcal{N}_q$-subset of (X, f). Since $() \leq y$, it follows that $(X, ()_\beta)$ is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f). But $f(()) > \frac{k-1}{2} > \beta$ implies that the point $\mathcal{N}$-structure $(X, ()_\beta)$ is not an $\mathcal{N}_\in$-subset of (X, f). Also $f(()) + \beta - k + 1 = \alpha_0 + \beta - k + 1 > 0$ implies that $(X, ()_\beta)$ is not an $\mathcal{N}_{q_k}$-subset of (X, f). This is a contradiction, and thus $f(y) \leq \frac{k-1}{2}$ for some $y \in X$. We now prove that $f(()) \leq \frac{k-1}{2}$. Assume that $\alpha_0 := f(()) > \frac{k-1}{2}$. Note that there exists $y \in X$ such that $\alpha_y := f(y) \leq \frac{k-1}{2}$ and so $\alpha_y < \alpha_0$. Choose $\alpha_1 < \alpha_0$ such that $\alpha_y + \alpha_1 - k + 1 < 0 < \alpha_0 + \alpha_1 - k + 1$. Then $f(y) + \alpha_1 - k + 1 = \alpha_y + \alpha_1 - k + 1 < 0$, and thus the point $\mathcal{N}$-structure (X, y_{α_1}) is an $\mathcal{N}_q$-subset of (X, f). Since $() \leq y$, we know that $(X, ()_{\alpha_1})$ is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f). But $f(()) + \alpha_1 - k + 1 = \alpha_0 + \alpha_1 - k + 1 > 0$ and also $f(()) = \alpha_0 > \alpha_1$ which is a contradiction. Therefore $f(()) \leq \frac{k-1}{2}$. ∎

Corollary 3.28 ([20]). *Let (X, f) be an $\mathcal{N}$-subnexus of type $(q, \in \vee q)$. If f is not constant on the open support of (X, f) related to 0 and $f(()) \geq f(x)$ for all $x \in X$, then there exists $y \in X$ such that $f(y) \leq -0.5$. In particular, $f(()) \leq -0.5$.*

Theorem 3.29. *If (X, f) is an $\mathcal{N}$-subnexus of type (q, q_k) such that $f(()) \geq f(x)$ for all $x \in X$, then f is constant on $O(f;k)$.*

Proof. Assume that f is not constant on $O(f;k)$. Then there exists $x \in O(f;k)$ such that $\alpha_x = f(x) \neq f(()) = \alpha_0$. Then $\alpha_0 > \alpha_x$, and so $f(x) + (-1 - \alpha_0) + 1 = \alpha_x - \alpha_0 < 0$. Hence $(X, x_{-1-\alpha_0})$ is an $\mathcal{N}_q$-subset of (X, f). Note that $() \leq x$ and $f(()) + (-1 - \alpha_0) - k + 1 = -k > 0$, which implies that $(X, ()_{-1-\alpha_0})$ is not an $\mathcal{N}_{q_k}$-subset of (X, f). This is a contradiction, and therefore f is constant on $O(f;k)$. ∎

Corollary 3.30 ([20]). *If (X, f) is an $\mathcal{N}$-subnexus of type (q, q) such that $f(()) \geq f(x)$ for all $x \in X$, then f is constant on the open support of (X, f) related to 0.*

4. $\mathcal{N}$-Subnexus of Type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$

Let (X, g) be an $\mathcal{N}$-structure. A point $\mathcal{N}$-structure (X, x_α) is said to be an $\mathcal{N}_{\overline{\in}}$-*subset* (resp., $\mathcal{N}_{\overline{q_k}}$-*subset*) of (X, g) if $g(x) > \alpha$ (resp., $g(x) + \alpha - k + 1 \geq 0$). If a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_{\overline{\in}}$-subset or an $\mathcal{N}_{\overline{q_k}}$-subset of (X, g), then (X, x_α) is said to be an $\mathcal{N}_{\overline{\in} \vee \overline{q_k}}$-*subset* of (X, g).

Definition 4.1. Let $v, w \in X$ such that $v \leq w$ and $\alpha \in [-1, 0)$. We say an $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-*subnexus* of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$, if (X, v_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f) then (X, w_α) is an $\mathcal{N}_{\overline{\in} \vee \overline{q_k}}$-subset of (X, f).

Similarly, we can define $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in})$. According to [20], we have (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in})$ if and only if it is an $\mathcal{N}$-subnexus of type $(\in, \in)$.

Moreover, it is important to note that every $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ for $k = 0$. Also, every $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$

is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$, for all $k \in (-1, 0]$, which is a different property with respect to other types of $\mathcal{N}$-subnexuses and their generalizations by $k \in (-1, 0]$. Therefore, according to [20], we can obtain the following corollaries:

Corollary 4.2. *Let (X, f) be an $\mathcal{N}$-structure such that for all $v, w \in X$ with $v \leq w$ we have $\bigwedge\{f(v), -0.5\} \leq f(w)$. Then, (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$.*

Corollary 4.3. *If the nonempty closed support of (X, f) related to α is a subnexus of X for every $\alpha \in [-1, -0.5)$, then (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$.*

Moreover, see the following example:

Example 4.4. Let (X, f) be an $\mathcal{N}$-structure in which $X = \{(), (1), (2), (2, 1), (2, 2)\}$ is a nexus and f is given as

$$f = \begin{pmatrix} () & (1) & (2) & (2,1) & (2,2) \\ -0.93 & -0.95 & -0.92 & -0.91 & -0.9 \end{pmatrix}.$$

We have $() \leq (1)$ and $f(()) = -0.93 > -0.94 = \alpha$. But $f((1)) < -0.94$ and $f(1) - 0.94 + 1 < 0$, and so (X, f) is not an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} vq)$. While, for $k = -0.9$ we have $f(1) - 0.94 - k + 1 \geq 0$ and therefore (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_{-0.9}})$.

Theorem 4.5. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ if and only if $\bigwedge\{f(v), \frac{k-1}{2}\} \leq f(w)$, for all $v, w \in X$ such that $v \leq w$.*

Proof. Let (X, f) be an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ and there exist $v, w \in X$ such that $v \leq w$ and $\bigwedge\{f(v), \frac{k-1}{2}\} > f(w) = \alpha$. So, $\alpha \in [-1, \frac{k-1}{2})$. It follows that (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f) and (X, v_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Hence (X, w_α) is an $\mathcal{N}_{\overline{q_k}}$-subset of (X, f). Therefore $2\alpha - k + 1 = f(w) + \alpha - k + 1 \geq 0$, which implies that $\alpha \geq \frac{k-1}{2}$. This is contradiction, and so $\bigwedge\{f(v), \frac{k-1}{2}\} \leq f(w)$ for all $v, w \in X$ with $v \leq w$. Conversely, let for all $v, w \in X$ such that $v \leq w$ we have $\bigwedge\{f(v), \frac{k-1}{2}\} \leq f(w)$. Let $v \leq w$ for $v, w \in X$ and $\alpha \in [-1, 0)$ such that a point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Then $f(v) > \alpha$. If $f(v) \leq f(w)$, then $\alpha < f(w)$ and thus (X, w_α) is an $\mathcal{N}_{\overline{\in} \vee \overline{q_k}}$-subset of (X, f). If $f(v) > f(w)$, then we have $f(w) \geq \bigwedge\{f(v), \frac{k-1}{2}\} = \frac{k-1}{2}$. Suppose that (X, w_α) is not an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Then $\alpha \geq f(w) \geq \frac{k-1}{2}$. Hence, we have $f(w) + \alpha - k + 1 \geq \frac{k-1}{2} + \frac{k-1}{2} - k + 1 = 0$ and so (X, w_α) is an $\mathcal{N}_{\overline{q_k}}$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$. ∎

Proposition 4.6. *If (X, f) is an $\mathcal{N}$-structure of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$, then $f(w) \geq f(())$ or $f(w) \geq \frac{k-1}{2}$ for all $w \in X$.*

Proof. Using Theorem 4.5, the proof is straightforward. ∎

Theorem 4.7. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ if and only if $\emptyset \neq C(f; \alpha)$ is a subnexus of X for every $\alpha \in [-1, \frac{k-1}{2})$.*

Proof. For an $\mathcal{N}$-subnexus (X, f) of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$, let $v, w \in X$ with $v \leq w$ and $w \in C(f; \alpha)$ for $\alpha \in [-1, \frac{k-1}{2})$. By Theorem 4.5, we have $\bigwedge\{f(v), \frac{k-1}{2}\} \leq f(w) \leq \alpha$. Since $\alpha < \frac{k-1}{2}$, then $f(v) \leq \alpha$ and so $v \in C(f; \alpha)$. Conversely, let (X, f) be an $\mathcal{N}$-structure such that $\emptyset \neq C(f; \alpha)$ is a subnexus of X for all $\alpha \in [-1, \frac{k-1}{2})$. Let for $v, w \in X$ such that $v \leq w$ we have $\bigwedge\{f(v), \frac{k-1}{2}\} > f(w)$. Put, $\beta := \frac{1}{2}\left(\bigwedge\{f(v), \frac{k-1}{2}\}\right) + f(w)$, then

$\beta \in [-1, \frac{k-1}{2})$ and $\bigwedge\{f(v), \frac{k-1}{2}\}) > \beta \geq f(w)$. Thus $w \in C(f;\beta)$ but $v \notin C(f;\beta)$ which is a contradiction. Therefore $\bigwedge\{f(v), \frac{k-1}{2}\}) \leq f(w)$ for all $v, w \in X$ with $v \leq w$. Using Theorem 4.5, then (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$. ∎

It is easy to see that every $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in})$ is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$, but the converse is not generally valid. See the following example:

Example 4.8. Let $X = \{(), (1), (2), (1,1), (1,2), (1,2,1)\}$ be a nexus and define h on X as

$$h = \begin{pmatrix} () & (1) & (2) & (1,1) & (1,2) & (1,2,1) \\ -0.4 & -0.45 & -0.33 & -0.3 & -0.34 & -0.2 \end{pmatrix}.$$

It is routine to verify that (X, h) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} vq)$ and so an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ for all $k \in (-1, 0]$. But it is not an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in})$ since $() \leq (1)$ and $(X, ()_{-0.42})$ is an $\mathcal{N}_{\overline{\in}}$-subset of (X, h), but $(X, (1)_{-0.42})$ is not an $\mathcal{N}_{\overline{\in}}$-subset of (X, h).

Theorem 4.9. *Let (X, f) be an $\mathcal{N}$-structure such that $f(x) \leq \frac{k-1}{2}$ for all $x \in X$. Then, every $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in})$.*

Proof. Let $v, w \in X$ and $\alpha \in [-1, 0)$ be such that $v \leq w$ and (X, v_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Then $f(v) > \alpha$. Since $f(x) \leq \frac{k-1}{2}$ for all $x \in X$, by Theorem 4.5, we have $\alpha < f(v) = \bigwedge\{f(v), \frac{k-1}{2}\} \leq f(w)$. Thus (X, w_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in})$. ∎

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