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Pursuit Differential Game Problem with Multiple Players on a Closed Convex Set with More General Integral Constraints

Idris Ahmed[1,2,3], **Wiyada Kumam**[4,*], **Gafurjan Ibragimov**[5] **and Jewaidu Rilwan**[1,2,6]

[1] *KMUTTFixed Point Research Laboratory, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*

[2] *Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Science Laboratory Building, King Mongkuts University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*

[3] *Department of Mathematics and Computer Science, Faculty of Natural and Applied Sciences, Sule Lamido University, P.M.B 048 Kafin-Hausa, Jigawa State, Nigeria*
e-mail : idrisahamedgml1988@gmail.com (I. Ahmed)

[4] *Program in Applied Statistics, Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Thanyaburi, Pathumthani 12110, Thailand*
e-mail : wiyada.kum@rmutt.ac.th (W. Kumam)

[5] *Institute for Mathematical Research and Department of Mathematics, Faculty of Science (FS), Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*
e-mail : ibragimov@upm.edu.my (G. Ibragimov)

[6] *Department of Mathematical Sciences, Faculty of Physical Sciences, Bayero University, Kano. P.M.B 3011 Kano, Nigeria*
e-mail : jrilwan.mth@buk.edu.ng (J. Rilwan)

Abstract In this paper, we study a pursuit differential game with many but finite number of pursuers and evaders on a closed convex subset K of $\mathbb{R}^n$. Players' motion obey ordinary differential equations and confined within the set K throughout the game. Control functions of the players are subject to general integral constraints respectively. We obtain sufficient conditions for completion of pursuit. Moreover, pursuers' strategies that ensure completion of pursuit in a finite time are constructed.

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*Corresponding author.

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1. Introduction

Differential game play an important role in numerous fields of studies such as engineering, control theory, missile guidance, behavioral biology and economics. Recently, its application in the field of fractional differential equations (FDEs) has received attention from researchers (see for example [1–3]). More details on the fundamental concepts of differential games can be found in books [4–6] are referred. Linear differential games with integral and geometric constraints on the control function of the players has also received considerable attention and fundamental results were obtained (see [7–23] and references therein). For Instance, Satimov et al. [24] studied the linear pursuit differential game of many pursuers and one evader with integral constraints on controls of players in $\mathbb{R}^n$, given by

$$\dot{z} = C_i z_i + u_i - v \quad z_i(t_0) = z_i^0, \quad i = 1, \ldots, m, \tag{1.1}$$

where u_i is the control parameter of the i^{th} pursuer and v is that of the evader. The eigenvalues of the matrices C_i are assumed to be real numbers. The above system (1.1), was extended to pursuit differential game of m pursuers and k evaders under the same integral constraints by Ibragimov et al. [25], with dynamic equations given by

$$\begin{gathered} \dot{z}_{ij} = C_{ij} z_{ij} + u_i - v_j, \quad z_{ij}(t_0) = z_{ij}^0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, k, \\ \int_0^\infty |u_i(s)|^2 ds \leq \rho_i^2, \quad i = 1, \ldots, m, \quad \int_0^\infty |v_j(s)|^2 ds \leq \sigma_j^2, \qquad j = 1, \ldots, k, \end{gathered} \tag{1.2}$$

where u_i is the control parameter of the i^{th} pursuer and v_j is that of the j^{th} evader. It is worth noting that the eigenvalues of the matrices C_i in (1.2) are not necessarily real and the number of evaders can be any. In [26], Ivanov generalized Lion and Man problem with respect to geometric constraints. All players have equal dynamic possibilities. They considered the following equations:

$$\dot{x_i} = u_i, \quad x_i(0) = x_{i0}, \quad |u_i| \leq 1, \quad i = 0, 1, \ldots, m, \tag{1.3}$$

where $x_i \in \mathbb{R}^n$, $u_i, i = 1, \ldots, m$, are control parameters of the pursuers and u_0 is control parameter of the evader, and the players may not leave a given compact subset N of $\mathbb{R}^n$. It was shown that if the number of pursuers m does not exceed the dimension of the space n, then evasion is possible; otherwise pursuit can be completed. The above system (1.3), was extended to game of many pursuers and evaders in [27].

Ibragimov [28] studied the differential game problem of one pursuer and one evader with integral constraints on a closed convex subset S of $\mathbb{R}^n$ given by

$$\begin{aligned} \dot{x}(t) &= \alpha(t)u(t), \quad x(0) = x_0, \quad \int_0^\infty |u(s)|^2 ds \leq \rho^2 \\ \dot{y}(t) &= \alpha(t)v(t), \quad y(0) = y_0, \quad \int_0^\infty |v(s)|^2 ds \leq \sigma^2, \end{aligned} \tag{1.4}$$

where the author proposed a formula for optimal pursuit time. Ibragimov and Satimov [29] extended the problem considered in [28] to pursuit differential game of m pursuers and k evaders on a closed convex subset of $\mathbb{R}^n$. They proved that pursuit can be completed from any initial position of the players under the condition that the energy resources of the pursuers is greater than that of the evaders.

Inspired by the paper of Ibragimov and Satimov [29] and some known results on pursuit differential game with integral and geometric constraints, this paper extend the results of [29] in which the control functions are choosing from the functional space L_p, and $\mathbb{R}^n$ is the state space which contains the closed convex set K.

2. Statement of Problem

Let $[0, \theta]$ be a finite interval on $\mathbb{R}$ and $C([0, \theta])$ be the space of all absolutely continuously differentiable function on $[0, \theta]$, respectively and $L_p(0, \theta)$, $(1 \leq p < \infty)$, q be such that $1/p + 1/q = 1$ and Lebesgue measurable functions defined on $(0, \theta)$. Let

$$\|f\|_{L_p(0,\theta)} = \left(\int_0^\theta \|f(t\|^p dt) \right)^{1/p} < \infty$$

for any $f \in L_p(0, \theta)$,

$$\|f\|_{C[0,\theta])} = Max\{\|f(t)\| : t \in [0, \theta]\}$$

for any $f \in C[0, \theta]$.
On the other hand, we define a set of absolutely continuous by

$$AC[0, \theta] = \{f : [0, \theta] \to \mathbb{R} : f^{n-1} \in AC[0, \theta]\}.$$

Consider a differential game problem with m pursuers and k evaders described by the equations

$$\begin{aligned} P_i &: \dot{x}_i(t) = \eta(t)u_i(t), \quad x_i(0) = x_{i0} \quad i = 1, 2, 3, \ldots, m, \\ E_j &: \dot{y}_j(t) = \eta(t)v_j(t), \quad y_j(0) = y_{j0}, \quad j = 1, 2, 3, \ldots, k, \end{aligned} \tag{2.1}$$

where $x_i(t), u_i(t), y_i(t), v_i(t) \in \mathbb{R}^n$, $u_i = (u_{i1}, \ldots, u_{in})$ is control parameter of the pursuer x_i, $v_j = (v_{j1}, \ldots, v_{jn})$ is that of the evader y_j, and the function $\eta(t)$ is a non zero in any open interval; scalar measurable and $q-integrable$ over any interval $[0, \tau]$, $\tau > 0$, and satisfies the following conditions:

$$a(\tau) = \left(\int_0^\tau \eta^q(t)dt \right)^{1/q}, \quad \lim_{\tau \to \infty} a(\tau) = \infty. \tag{2.2}$$

Definition 2.1. A measurable function $u_i(t) = (u_{i1}(t), ..., u_{in}(t))$, $t \geq 0$, is called an admissible control of the pursuer x_i if $x_i(t) \in K$, $t \geq 0$, and

$$\int_0^\infty |u_i(s)|^p ds \leq \rho_i^p, \tag{2.3}$$

where

$$x_i(t) = x_{i0} + \int_0^t \eta(s)u_i(s)ds,$$

$\rho_i, i = 1, \ldots, m$, are given positive numbers. We denote the set of all admissible controls of the pursuer x_i by $S(\rho_i)$.

Definition 2.2. A measurable function $v_j(t) = (v_{j1}(t), ..., v_{jn}(t))$, $t \geq 0$, is called an admissible control of the evader y_j, if $y_j(t) \in K$, $t \geq 0$, and

$$\int_0^\infty |v_j(s)|^p ds \leq \sigma_j^p, \tag{2.4}$$

where

$$y_j(t) = y_{j0} + \int_0^t \eta(s)v_j(s)ds,$$

$\sigma_j, j = 1, \ldots, m$, are given positive numbers. We denote the set of all admissible controls of the evader y_j by $S(\sigma_j)$.

Definition 2.3. A Borel measurable function $U_i = U_i(x_i, y_1, ...y_k, v_1, ..., v_k), U_i : \mathbb{R}^{2k+1} \to \mathbb{R}^n$, is called a strategy of the pursuer x_i if for any controls of the evaders $v_j(t) \quad j = 1, \ldots, k, \quad t \geq 0$, the initial value problem

$$\begin{aligned} \dot{x_i} &= \eta(t)U_i(x_i, y_1, ...y_k, v_1(t), ..., v_k(t)) \quad x_i(0) = x_{i0}, \\ \dot{y_j} &= \eta(t)v_j(t), \quad y_j(0) = y_{j0}, \quad j = 1, ...k, \end{aligned} \tag{2.5}$$

has a unique absolutely continuous solution $(x_i(t), y_1(t), ..., y_k(t))$ and along this solution the following inequality

$$\int_0^\infty |U_i(x_i(t), y_1(t), ...y_k(t), v_1(t), ..., v_k(t))|^p dt \leq \rho_i^p \tag{2.6}$$

holds, and $x_i \in K, t \geq 0$.

Definition 2.4. Pursuit is said to be completed from the initial positions $\{x_{10}, \cdots, x_{m0}, y_{10}, \cdots, y_{k0}\}$ at the time T in the game described by (2.1)-(2.4), if there exist strategies U_i, $i = 1, ...m$, of the i^{th} pursuers such that for any controls $v_1(\cdot), ..., v_k(\cdot)$ of the evaders and numbers $j = 1, 2, ...k$, the equality $x_i(t_j) = y_j(t_j)$ holds for some $i \in \{1, ..., m\}$ at some time $t_j \in [0, \ T]$.

It is also required that all players must not leave a nonempty convex subset C of $\mathbb{R}^n$, throughout the game. This means that

$$x_{i0}, x_i(t), y_{j0}, y_j(t) \in K,$$

for all $t \geq 0$, and for all $i = 1, ..., m, j = 1, ..., k$.

The problem is to find sufficient conditions for the completion of pursuit in the game described by (2.1)-(2.4).

3. Main Result

The following statement gives sufficient condition for completion of pursuit.

Theorem 3.1. *For pursuit to be completed in the game described by* (2.1)-(2.4) *from any initial position of the players in a finite time T, it is sufficient that*

$$\sum_{i=1}^{m} \rho_i^p > \sum_{j=1}^{k} \sigma_j^p, \quad \text{and} \quad \sigma > 1. \tag{3.1}$$

Proof.

1. **Auxiliary game**: To prove this theorem, first we study the game problem with only one pursuer and one evader, which means $m = k = 1$. That is, we study the game described by

$$\begin{aligned} P &: \dot{x} = \eta(t)u, \quad x(0) = x_0, \\ E &: \dot{y} = \eta(t)v, \quad y(0) = y_0, \end{aligned} \tag{3.2}$$

where $x(t), y(t) \in \mathbb{R}^n$, u is the control function of the pursuer x, and v is that of the evader y. Assume that $u = u(\cdot) \in S(\rho)$, $v = v(\cdot) \in S(\sigma)$. In this game, we allow the players to move freely in the space $\mathbb{R}^n$. We say pursuit is completed if $x(\theta) = y(\theta)$ at some time $\theta \geq 0$.

The following lemma provides the condition that ensure completion of pursuit in a finite time θ in the game described by (3.2), where θ is an arbitrary fixed number satisfying the inequality

$$a(\theta) \geq \frac{|y_0 - x_0|}{\rho - \sigma}. \tag{3.3}$$

Let us now denote the following,

$$\rho^p(t) = \rho^p - \int_0^t |u(s)|^p ds, \quad \sigma^p(t) = \sigma^p - \int_0^t |v(s)|^p ds, \tag{3.4}$$

We will now prove the following statement.

Lemma 3.2. *(i). If $\rho > \sigma$, then pursuit can be completed in the game described by (3.2) for the time θ. Furthermore,*

$$\rho^p(\theta) \geq \rho^p - \sigma^p - \frac{|x_0 - y_0|}{a(\theta)}. \tag{3.5}$$

(ii). If $\rho \leq \sigma$, then either $x(\theta) = y(\theta)$ or

$$\sigma^p(\theta) < \sigma^p - \rho^p + \frac{|x_0 - y_0|}{a(\theta)}. \tag{3.6}$$

Proof. Let us construct the strategy of the pursuer as follows

$$u(t) = \frac{\eta^{q-1}(t)}{a^q(\theta)}(y_0 - x_0) + v(t). \tag{3.7}$$

Show that the strategy (3.7) is admissible and ensures the equality $x(\theta) = y(\theta)$. Indeed, clearly

$$\begin{aligned} x(\theta) &= x_0 + \int_0^\theta \eta(t) \left(\frac{\eta^{q-1}(t)}{a^q(\theta)}(y_0 - x_0) + v(t) \right) dt \\ &= x_0 + \frac{(y_0 - x_0)}{a^q(\theta)} \int_0^\theta \eta^q(t) dt + \int_0^\theta \eta(t) v(t) dt \\ &= y_0 + \int_0^\theta \eta(t) v(t) dt = y(\theta). \end{aligned} \tag{3.8}$$

To show the admissibility of the strategy (3.7), we use the Minkowski's inequality, thus

$$\begin{aligned} \int_0^\theta |u(t)|^p dt &= \int_0^\theta \left| \left(\frac{\eta^{q-1}(t)}{a^q(\theta)}(y_0 - x_0) + v(t) \right) \right|^p dt \\ &\leq \left(\int_0^\theta \left(\frac{\eta^{q-1}(t)}{a^q(\theta)} |y_0 - x_0| \right)^p dt \right)^{1/p} + \left(\int_0^\theta |v(t)|^p dt \right)^{1/p} \\ &\leq \frac{|y_0 - x_0|}{a^q(\theta)} a^{q-1}(\theta) + \sigma \\ &\leq \frac{(\rho - \sigma)}{|y_0 - x_0|} |y_0 - x_0| + \sigma = \rho. \end{aligned} \tag{3.9}$$

Thus, we have proved the admissibility of the strategy (3.7).
To show inequality (3.5), we proceed as follows:
From (3.4) and (3.9), we obtain

$$\begin{aligned}\int_0^\theta |u(t)|^p dt &\leq \frac{|y_0 - x_0|}{a(\theta)} + \sigma \\ &\leq \frac{|x_0 - y_0|}{a(\theta)} + \sigma^p,\end{aligned} \tag{3.10}$$

this implies

$$\rho^p(\theta) \geq \rho^p - \sigma^p - \frac{|y_0 - x_0|}{a(\theta)}. \tag{3.11}$$

Thus, we have the required inequality (3.5). We now proof part (ii) of the Lemma (3.2). Let $\rho \leq \sigma$ and the pursuer use the strategy (3.7) on the interval $[0, \theta]$, for any control of the evader $v(t), t \in [0, \theta]$, the inequality is satisfied:

$$\int_0^\theta |u(t)|^p dt = \int_0^\theta \left| \left(\frac{\eta^{q-1}(t)}{a^q(\theta)}(y_0 - x_0) + v(t) \right) \right|^p dt \leq \rho^p, \tag{3.12}$$

therefore, the strategy (3.7) is admissible, and similar to (3.8) we obtain $x(\theta) = y(\theta)$, hence the proof of the lemma fallows. Suppose that $x(\theta) \neq y(\theta)$, then we must have

$$\int_0^\theta |u(t)|^p dt = \int_0^\theta \left| \left(\frac{\eta^{q-1}(t)}{a^q(\theta)}(y_0 - x_0) + v(t) \right) \right|^p dt > \rho^p. \tag{3.13}$$

Thus, from inequality (3.13), and using the same calculations to (3.9), we obtain

$$\begin{aligned}\int_0^\theta |u(t)|^p dt &\leq \frac{|y_0 - x_0|}{a(\theta)} + \left(\int_0^\theta |v(t)|^p dt \right)^{1/p} \\ &\leq \frac{|y_0 - x_0|}{a(\theta)} + \int_0^\theta |v(t)|^p dt\end{aligned} \tag{3.14}$$

and therefore

$$\sigma^p(\theta) = \sigma^p - \int_0^\theta |v(t)|^p dt < \sigma^p - \rho^p + \frac{|y_0 - x_0|}{a(\theta)}. \tag{3.15}$$

Hence, the proof of the lemma is complete. ∎

2. **Dummy pursuers (DPs)**: To prove the main theorem, we introduce dummy pursuers $z_1, \ldots, z_m$ whose equations of motion are described by

$$\dot{z}_i = \eta(t) w_i, \quad z_i(0) = x_{i0}, \quad w_i \in S(\rho_i), \quad i = 1, ..., m, \tag{3.16}$$

where w_i is control parameter of the pursuer z_i. Dummy pursuers may move out of the convex set $K \subset \mathbb{R}^n$. The aim of the dummy pursuer is to complete the pursuit in a finite time.
We define the dummy pursuers strategies in the time interval $[0, \theta_1]$ as follows:

$$w_m(t) = \frac{\eta^{q-1}(t)}{a^q(\theta_1)}(y_{k0} - x_{m0}) + v_k(t), \quad 0 \leq t \leq \theta_1, \tag{3.17}$$

$$w_i(t) \equiv 0, \quad i = 1, ..., m-1, \quad 0 \leq t \leq \theta_1, \tag{3.18}$$

where θ_1 is any number satisfying inequalities

$$a(\theta_1) \geq \frac{|y_{k0} - x_{m0}|}{(\rho_m - \sigma_k)}, \tag{3.19}$$

$$\sum_{i=1}^{m} \rho_i^p > \sum_{j=i}^{k} \sigma_j^p + \frac{|y_0 - x_0|}{a(\theta_1)}. \tag{3.20}$$

Since by assumption $a(t) \to \infty$ as $t \to \infty$, it follows from (3.1) that such θ_1 exists. Equations (3.17) and (3.18) mean that all dummy pursuers, $z_1, \ldots, z_{m-1}$ do not move on the time interval $[0, \theta_1]$, and only one dummy pursuer, z_m, moves according to (3.17).
We will now show that if inequality (3.1) holds and the dummy pursuers use the strategies (3.17) and (3.18), then the pursuit problem with the pursuers $z_1, \ldots, z_m$ and evaders $y_1, \ldots, y_k$ is reduced to a pursuit problem with the pursuers $z_1, \ldots, z_a$ and evaders $y_1, \ldots, y_b$, for which

$$\sum_{i=1}^{a} \rho_i^p(\theta_1) > \sum_{j=1}^{b} \sigma_j^p(\theta_1) \tag{3.21}$$

and $a + b < m + k$. Hence, by the time θ_1, the number of players is reduced to $a + b$.
Indeed, in the first phase of the game, if the dummy pursuers use the strategy (3.17) and (3.18), and if $\rho_m \leq \sigma_k$, and the equality $z_m(\theta_1) = y_k(\theta_1)$ holds then the m^{th} *pursuer* will catch the k^{th} *evader*. Therefore the number of pursuers and evaders is reduce to $m - 1$ and $k - 1$, respectively.
Now a game problem with pursuers $z_1, \ldots, z_{m-1}$ and evaders $y_1, \ldots, y_{k-1}$ is considered with the condition

$$\sum_{i=1}^{m-1} \rho_i^p > \sum_{j=1}^{k-1} \sigma_j^p \quad (a = m - 1, b = k - 1). \tag{3.22}$$

If $z_m(\theta_1) \neq y_k(\theta_1)$, according to (3.6), we obtain that

$$\sigma_k^p(\theta_1) \leq \sigma_k^p - \rho_m^p + \frac{|x_0 - y_0|}{a(\theta_1)}. \tag{3.23}$$

Then, in view of (3.20) and (3.23), we obtain

$$\sum_{i=1}^{m-1} \rho_i^p > \sum_{j=1}^{k} \sigma_j^p \quad (a = m - 1, b = k), \tag{3.24}$$

and at the time θ_1, we consider the pursuit problem with the pursuers $z_1, \ldots, z_{m-1}$ and evaders $y_1, \ldots, y_k$. We now turn to the second phase of the game, that is if $\rho_m > \sigma_k$. Then the pursuer z_m ensures the equality $z_m(\theta_1) = y_k(\theta_1)$ and according to lemma 3.2 (see, (3.5))

$$\rho_m^p(\theta_1) \geq \rho_m^p - \sigma_k^p - \frac{|y_0 - x_0|}{a(\theta_1)}. \tag{3.25}$$

Then from equations (3.20) and (3.25), we obtain

$$\sum_{i=1}^{m-1} \rho_i^p + \rho_m^p(\theta_1) > \sum_{j=1}^{k-1} \sigma_j^p \quad (a = m, b = k - 1), \tag{3.26}$$

and therefore at the time θ_1 we considered the pursuit problem with the pursuers $z_1, \ldots, z_m$ and evaders $y_1, \ldots, y_{k-1}$.

Let θ_2 be an arbitrary fixed number satisfying inequalities

$$\begin{cases} \theta_2 > \theta_1, \\ a(\theta_2 - \theta_1) \geq \frac{|y_b(\theta_1) - z_l(\theta_1)|}{(\rho_l(\theta_1) - \sigma_b(\theta_1))}, \\ \rho_1^p(\theta_1) + \cdots + \rho_l^p(\theta_1) > \sigma_1^p(\theta_1) + \cdots + \sigma_b^p(\theta_1) + \frac{|y_0, x_0|}{a(\theta_2 - \theta_1)}, \end{cases} \tag{3.27}$$

where l and b are the numbers of pursuers and evaders, respectively, which take part in the pursuit problem at time θ_1. Define the pursuers strategy as follows:

$$\begin{aligned} w_a(t) &= \frac{\eta(t)^{q-1}}{a^q(\theta_2 - \theta_1)}(y_b(\theta_1) - z_l(\theta_1)) + v_b(t), \quad \theta_1 < t \leq \theta_2, \\ w_i &\equiv 0, \quad \theta_1 < t \leq \theta_2, \quad i \in \{1, ..., m\} \backslash \{l\}. \end{aligned} \tag{3.28}$$

Observe that according to (3.28), all pursuers except z_l will not move on the time interval $(\theta_1, \theta_2]$. Let the pursuers use the strategies (3.28). Applying the same arguments above, we arrive at the following:

(i) If $\rho_l(\theta_1) \leq \sigma_b(\theta_1)$ and $z_l(\theta_2) = y_b(\theta_2)$, then starting from θ_2 we consider a pursuit problem with the pursuers $z_1, \ldots, z_{l-1}$ and evaders $y_1, ..., y_{b-1}$ under the condition

$$\sum_{i=1}^{l-1} \rho_i^p(\theta_2) > \sum_{j=1}^{b-1} \sigma_j^p(\theta_2). \tag{3.29}$$

(ii) If $\rho_l(\theta_1) \leq \sigma_b(\theta_1)$ and $z_l(\theta_2) \neq y_b(\theta_2)$, then starting from θ_2 we consider a pursuit problem with the pursuers $z_1 ..., z_{l-1}$ and evaders $y_1, \ldots, y_b$ under the condition

$$\sum_{i=1}^{l-1} \rho_i^p(\theta_2) > \sum_{j=1}^{b} \sigma_1^p(\theta_1). \tag{3.30}$$

(iii) If $\rho_a(\theta_1) > \sigma_b(\theta_1$, then by lemma 3.2 the equality $z_l(\theta_2) = y_b(\theta_2)$ holds. In this case, starting from θ_2 the pursuit problem with the pursuers $z_1, \ldots, z_l$ and evaders $y_1, \ldots, y_{b-1}$ is considered under the condition

$$\sum_{i=1}^{l} \rho_i^p(\theta_2) > \sum_{j=1}^{b-1} \sigma_1^p(\theta_2). \tag{3.31}$$

Repeated application of this procedure enables us to complete the pursuit for some finite time T since the number of players is finite and decreasing. Thus, we have proved that dummy pursuers can complete the pursuit.

3. **Completion of the proof of the theorem**: We will now show that the actual pursuers also can complete the pursuit. Define the controls $u_1, \cdots, u_m$ by the control of the DPs $w_1, \cdots, w_m$. Let $P_K(x)$ denote the projection of the point $x \in \mathbb{R}^n$ on the set K, (where K is a closed convex subset of $\mathbb{R}^n$), then we obtain the following results. Note that $P_K(x) = x$ if $x \in \mathbb{R}^n$, the we have

$$|x - y| \leq |x - P_K(x)|, \tag{3.32}$$

and

$$|P_K(x) - P_K(y)| \leq |x - y|. \tag{3.33}$$

Hence the operator $P_K(x)$ relates any absolute continuous function $z(t)$, $\quad 0 \leq t \leq T$, to an absolute continuous function

$$x(t) = P_K(z(t), \quad 0 \leq t \leq T, \tag{3.34}$$

where T is the time in which pursuit can be completed by DPs. We construct the strategy for the actual pursuers to satisfy

$$x_i(t) = P_K(z_i(t)), \quad 0 \leq t \leq T, \quad i \in \{1, \cdots, m\}. \tag{3.35}$$

We now show that the strategy (3.35), is admissible. Indeed from (3.32), (3.33), and (3.35), we have almost everywhere on $[0, T]$, that is

$$\begin{aligned} \eta(t)|u_i(t)| &= |\dot{x}_i(t)| \\ &= \lim_{k \to 0^+} \left| \frac{x_i(t+k) - x(t)}{k} \right| \\ &= \lim_{k \to 0^+} \frac{|P_C(z(t+k)) - P_C(z(t))|}{|k|} \\ &\leq \lim_{k \to 0^+} \frac{|z_i(t+k) - z_i(t)|}{|k|} \\ &= |\dot{z}(t)| \\ &= \eta(t)|w_i(t)|dt. \end{aligned} \tag{3.36}$$

This implies

$$\int_0^T |u_i(t)|^p dt \leq \int_0^T |w_i(t)|^p dt \tag{3.37}$$

and therefore at some time $t_i \leq T$ and $n_i \in \{1, \cdots, m\}$, the equality

$$z_{ni}(t_i) = y_i(t_i). \tag{3.38}$$

holds for any evader $y_i, i \in \{1, \cdots, m\}$. In particular, $y_i(t_i) \in C$, by (3.35), and the fact that $P_K(x) = x$ if $x \in \mathbb{R}^n$, we have

$$x_{ni}(t_i) = P_K(z_{ni}(t_i)) = z_{ni}(t_i) = y_i(t_i). \tag{3.39}$$

This means the differential game can be completed for the time T. therefore the proof of the theorem is complete. ■

Corollary 3.3. *If $p = q = 2$, condition (3.1) of the theorem reduces to $\rho_1^2 + \cdots + \rho_m^2 \geq \sigma_2^2 + \cdots + \sigma_k^2$, and therefore pursuit can be completed [29].*

4. Conclusions

We have studied a pursuit differential game of many but finite number of pursuers and evaders on a closed convex subset K of $\mathbb{R}^n$. The main contribution of this paper (Theorem 3.1) is that, we extend the result of [29] in the space $\mathbb{R}^n$ to the functional space L_p. That is, if $p = q = 1$, the condition (3.1) of Theorem 3.1 takes the form $\rho_1^2 + \cdots + \rho_m^2 \geq \sigma_2^2 + \cdots + \sigma_k^2$, then pursuit can be completed.

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