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Coincidence Point Theorems for *BKC*-Contraction Mappings in Generalized Metric Spaces Endowed with a Directed Graph

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Abstract A new concept of generalized metric spaces and BKC-contractions are proposed while the existence of coincident points in these new settings is considered. In addition, an example as well as an application are provided to support our results.

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1. Introduction

Generalized metric spaces are extensively investigated in fixed point theory. This is due to the fact that such spaces recover an abundant classes of previous studied spaces. Some of outstanding examples of generalized metric spaces are suggested by Jleli and Samet via the notion of JS-metric spaces [1] which are widely interested among researchers in the field (see [2–6] for example).

In our work, we are focusing on generalized metric spaces endowed with a graph. The idea of metric spaces endowed with a graph is first originated by Jachymski [7]. Since then, many authors have studied the problem of existence of a fixed point for single-valued mappings and multi-valued mappings in several spaces with a graph, for instance, see [8–11]. In particular, we introduce a generalization of the famous Banach contractions, and Kannan and Chatterjea contractions, namely BKC-contractions in order to examine existence theorems in this more general framework.

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2. Main Results

Let us provide definitions and background knowledge which are useful throughout our work.

First, we give the notion of limits of sequences.

Definition 2.1. Given a nonempty set X and a function $D : X \times X \to [0, +\infty]$, for any $x \in X$, we define

$$C(D, X, x) = \{\{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0\}$$

where $\{x_n\} \subseteq X$ means that $\{x_n\}$ is a sequence in X. In addition, we also write $\lim_{n\to\infty} x_n = x$ whenever $\{x_n\} \in C(D, X, x)$.

Next, we define generalized metric spaces as follows.

Definition 2.2. Suppose X is a nonempty set. A function $D : X \times X \to [0, +\infty]$ will be called a **generalized metric** on X if the following conditions hold.

(D_1) For every $x, y \in X$, if $D(x, y) = 0$, then we have $x = y$;
(D_2) For every $x, y \in X$, we have $D(x, y) = D(y, x)$; and
(D_3) There is a positive number $C_X > 0$ which satisfies

$$D(x, y) \leq C_X \limsup_{n\to\infty} D(x_n, y_n)$$

for every $x, y \in X$ such that $\{x_n\} \in C(D, X, x)$ and $\{y_n\} \in C(D, X, y)$.

In this situation, we will say that (X, D) is a **generalized metric space**.

Example 2.3. Consider $X = [0, 1]$ and a function $D : X \times X \to [0, +\infty]$ such that

$$D(x, y) = \begin{cases} 2(x + y), & x \neq 0 \text{ and } y \neq 0, \\ x, & y = 0, \\ y, & x = 0. \end{cases}$$

It is not hard to see that (X, D) becomes a generalized metric space.

Definition 2.4. Given that (X, D) is a generalized metric space and $\{x_n\} \subseteq X$, we would say $\{x_n\}$ **D-converges** to $x \in X$ if $\{x_n\} \in C(D, X, x)$. In addition, $\{x_n\}$ is **D-Cauchy** whenever $\lim_{m,n\to\infty} D(x_n, x_m) = 0$. Moreover, (X, D) is **D-complete** whenever each D-Cauchy sequence in X is D-converging to an element of X.

Definition 2.5. Given a generalized metric space (X, D), a function $f : X \to X$ is said to be **continuous at a point** $x_0 \in X$ whenever $\{x_n\} \in C(D, X, x_0)$ implies $\{fx_n\} \in C(D, X, fx_0)$. If f is continuous at each x in X, then f is called **continuous**.

Definition 2.6. Assume that (X, D) is a generalized metric space; G is a directed graph such that the set of vertices, $V(G)$, is exactly X; and $\Delta \subseteq E(G)$, where Δ and $E(G)$ denote the diagonal of $X \times X$ and the set of edges of G, respectively. Suppose further that G does not have parallel edges so that G can be identified with $(V(G), E(G))$.

We say that a (X, D) **is endowed with a directed graph** G if and only if (X, D) satisfies all of the above assumptions.

Definition 2.7. Given that (X, D) is a generalized metric space endowed with a directed graph G, a function $f : (X, D) \to (X, D)$ is said to be **G-continuous** whenever for any $x \in X$ which has $\{x_n\} \in C(D, X, x)$ such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we obtain $\{fx_n\} \in C(D, X, fx)$.

Definition 2.8. Given that (X, D) is a generalized metric space endowed with a directed graph G, and $f, g : X \to X$ are functions, we will say that f is **g-edge preserving w.r.t G** if

$$(gx, gy) \in E(G) \text{ implies } (fx, fy) \in E(G).$$

Definition 2.9. Given that (X, D) is a generalized metric space endowed with a directed graph G, we will say that $E(G)$ satisfies the **transitivity property** if and only if for any $(x, y, z) \in X \times X \times X$, we get

$$(x, z), (z, y) \in E(G) \text{ implies } (x, y) \in E(G).$$

Definition 2.10. Assume that (X, D) is a generalized metric space endowed with a directed graph G, and $f, g : X \to X$ are functions. We define the **set of all coincidence points of f and g** by

$$C(f, g) = \{u \in X : fu = gu\}.$$

Furthermore, for each sequence $\{x_n\} \subseteq X$ and $n \in \mathbb{N} \cup \{0\}$, we define

$$\beta(D, f, x_n) = \sup\{D(fx_{n+i}, fx_{n+j}) : i, j \in \mathbb{N})\},$$

and

$$X(f, g) = \{x_0 \in X : (gx_0, fx_0) \in E(G) \text{ and } \beta(D, f, x_0) < \infty\}.$$

Definition 2.11. Given a generalized metric space (X, D), and functions $f, g : X \to X$, we say that f and g are **D-compatible** if and only if

$$\lim_{n\to\infty} D(gfx_n, fgx_n) = 0$$

for all $\{gx_n\} \in C(D, X, a)$ and $\{fx_n\} \in C(D, X, a)$, where $a \in X$.

In the following definition, we define a generalization of Banach contractions, and Kannan and Chatterjea contractions.

Definition 2.12. Given that (X, D) is a generalized metric space endowed with a directed graph G, and $f, g : X \to X$ are functions, the pair (f, g) is said to be a **BKC-contraction** if the following hold.

(i) f is *g-edge preserving w.r.t G*; and
(ii) There is $\lambda \in [0, 1/2)$ such that for any $x, y \in X$ with $(gx, gy) \in E(G)$, we receive

$$D(fx, fy) \leq \lambda \max\{2D(gx, gy), D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\}. \tag{2.1}$$

We start our results with the following theorem about the existence of coincidence points in the case of BKC-contractions.

Theorem 2.13. *Assume that (X,D) is a D-complete generalized metric space endowed with a directed graph G, and $f,g : X \to X$ are functions such that (f,g) is a BKC-contraction. Furthermore, assume that the following hold.*

(a) $f(X) \subseteq g(X)$;
(b) $E(G)$ satisfies the transitivity property;
(c) There is $x_0 \in X(f,g)$;
(d) f is G-continuous while g is continuous; and
(e) f and g are D-compatible.

Then it is true that $C(f,g) \neq \emptyset$.

Proof. By the assumption, we obtain that $(gx_0, fx_0) \in E(G)$ and $\beta(D,f,x_0) < \infty$. Furthermore, $f(X) \subseteq g(X)$ implies that we can construct a sequence $\{x_n\} \subseteq X$ such that for each $n \in \mathbb{N}$,

$$gx_n = fx_{n-1}.$$

Suppose that $gx_{n_0} = gx_{n_0-1}$ for some $n_0 \in \mathbb{N}$. Then it is obvious that x_{n_0-1} becomes a coincidence point of f and g. Hence, it is worth considering only the case which $gx_n \neq gx_{n-1}$ for any $n \in \mathbb{N}$.

Because $(gx_0, fx_0) = (gx_0, gx_1) \in E(G)$ and f is g-edge preserving w.r.t G, we get that $(fx_0, fx_1) = (gx_1, gx_2) \in E(G)$. Continuing these steps inductively, we receive

$$(gx_n, gx_{n+1}) \in E(G) \quad \text{for each} \quad n \in \mathbb{N}. \tag{2.2}$$

Because $E(G)$ satisfies the transitivity property, we also get

$$(gx_k, gx_l) \in E(G) \quad \text{for each} \quad k,l \in \mathbb{N} \ \text{ such that } \ k < l. \tag{2.3}$$

Now, for each $n \in \mathbb{N}$ with $n \geq 2$, and $i,j \in \mathbb{N}$, we obtain that

$$\begin{aligned} D(gx_{n+i+1}, gx_{n+j+1}) &= D(fx_{n+i}, fx_{n+j}) \\ &\leq \lambda \max\big\{2D(gx_{n+i}, gx_{n+j}), D(gx_{n+i}, fx_{n+i}) + D(gx_{n+j}, fx_{n+j}), \\ &\qquad D(gx_{n+i}, fx_{n+j}) + D(gx_{n+j}, fx_{n+i})\big\} \\ &\leq 2\lambda\beta(D,f,x_{n-1}). \end{aligned}$$

Therefore,

$$\beta(D,f,x_n) \leq 2\lambda\beta(D,f,x_{n-1})$$

and hence

$$\beta(D,f,x_n) \leq (2\lambda)^n\beta(D,f,x_0).$$

Then we have

$$D(gx_n, gx_m) = D(fx_{n-1}, fx_{m-1}) \leq \beta(D,f,x_{n-2}) \leq (2\lambda)^{n-2}\beta(D,f,x_0)$$

for every integer m with $m > n$.

By the fact that $\beta(D,f,x_0) < \infty$ and $2\lambda < 1$, we obtain

$$\lim_{n,m\to\infty} D(gx_n, gx_m) = 0$$

which implies that $\{gx_n\}$ is D-Cauchy.

On the other hand, because (X,D) is D-complete, there will be $u \in X$ such that

$$\lim_{n\to\infty} D(gx_n, u) = \lim_{n\to\infty} D(fx_n, u) = 0.$$

As a result,

$$\{gx_n\}, \{fx_n\} \in C(D, X, u).$$

In addition, the G-continuity of f and the continuity of g provide

$$\{fgx_n\} \in C(D, X, fu) \quad \text{and} \quad \{gfx_n\} \in C(D, X, gu).$$

By property (D_3) and the fact that f and g are D-compatible, we get

$$D(gu, fu) \leq C_X \limsup_{n\to\infty} D(gfx_n, fgx_n) = 0.$$

So $fu = gu$ and we can conclude that u is a coincidence point of f and g. ■

Let us provide some observations about the properties of BKC-contractions in the following proposition.

Proposition 2.14. *Given that (X, D) is a generalized metric space endowed with a directed graph G, and $f, g : X \to X$ are functions such that (f, g) is a BKC-contraction, we have that for any $x, y \in C(f, g)$,*

(a) if $D(gx, gx) < \infty$, then $D(gx, gx) = 0$; and
(b) if $D(gx, gy) < \infty$ and $(gx, gy) \in E(G)$, then $gx = gy$.

Proof. (1) Given $x \in C(f, g)$, we have

$$\begin{aligned} D(gx, gx) &= D(fx, fx) \\ &\leq \lambda \max\{2D(gx, gx), D(gx, fx) + D(gx, fx), D(gx, fx) + D(gx, fx)\} \\ &\leq 2\lambda D(gx, gx). \end{aligned}$$

This implies $D(gx, gx) = 0$ since $2\lambda < 1$.

(2) Given $x, y \in C(f, g)$ and $(gx, gy) \in E(G)$, we obtain

$$\begin{aligned} D(gx, gy) &= D(fx, fy) \\ &\leq \lambda \max\{2D(gx, gy), D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\}. \end{aligned}$$

Let us consider the following 3 cases which will provide the desired result.

If $\max\{2D(gx, gy), D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\} = 2D(gx, gy)$, we have

$$D(gx, gy) \leq 2\lambda D(gx, gy).$$

Since $2\lambda \in [0, 1)$, $gx = gy$.

If $\max\{2D(gx,gy), D(gx,fx)+D(gy,fy), D(gx,fy)+D(gy,fx)\} = D(gx,fx)+D(gy,fy)$, we have

$$\begin{aligned} D(gx, gy) &= D(fx, fy) \\ &\leq \lambda\{D(gx, fx) + D(gy, fy)\} \\ &= \lambda\{D(gx, gx) + D(gy, gy)\} \\ &= 0 \end{aligned}$$

which means that $gx = gy$.

If $\max\{2D(gx,gy), D(gx,fx)+D(gy,fy), D(gx,fy)+D(gy,fx)\} = D(gx,fy)+D(gy,fx)$, we have

$$\begin{aligned} D(gx,gy) &= D(fx,fy) \\ &\leq \lambda\{D(gx,fy)+D(gy,fx)\} \\ &= \lambda\{D(gx,gy)+D(gy,gx)\} \\ &= 2\lambda\{D(gx,gy)\}. \end{aligned}$$

Since $\lambda < 1/2$, we obtain that $D(gx,gy) = 0$ and then $gx = gy$. ∎

Definition 2.15. Given that (X,D) is a generalized metric space endowed with a directed graph G, and $f,g : X \to X$ are functions, we define the **set of all common fixed points of mappings f and g** by

$$Cm(f,g) = \{u \in X : fu = gu = u\}.$$

Theorem 2.16. *In addition to the hypotheses of Theorem 2.13, assume further that*

(f) $(gx,gy) \in E(G)$ for any $x,y \in C(f,g)$, and

(g) f and g are commuting.

Then we have $Cm(f,g) \neq \emptyset$.

Proof. From Theorem 2.13, we obtain that $C(f,g) \neq \emptyset$. Now, let $u \in C(f,g)$ such that $c = gu = fu$. Since f and g are commuting, $gc = gfu = fgu = fc$. Thus, $c \in C(f,g)$. By the assumption (f), we have $(gu,gc) \in E(G)$. Then, by proposition 2.14, we can conclude that $fc = gc = gu = c$. Hence, $c \in Cm(f,g)$ and the proof is complete. ∎

Example 2.17. Let $X = [0,1]$ and let D be a generalized metric such that

$$D(x,y) = \begin{cases} 2(x+y), & x \neq 0 \text{ and } y \neq 0, \\ x, & y = 0, \\ y, & x = 0. \end{cases}$$

Then it is not hard to see that $C(D,X,a) \neq \varnothing$ if and only if $a = 0$ and (X,D) is D-complete.

Let us define

$$E(G) = \{(x,y) : x \neq 0 \text{ or } y = 0\}.$$

Furthermore, let $f,g : X \to X$ be defined by

$$f(x) = \frac{x}{2x+24} \quad \text{and} \quad g(x) = \frac{x}{8}.$$

We will use Theorem 2.13 to assert that $C(f,g) \neq \emptyset$.

To start with, it can be checked that $f(X) \subseteq g(X)$ and $g(X)$ is D-complete because $0 \in g(X)$. Next, we choose $x_0 = 1$ so that $(g(1), f(1)) = (\frac{1}{8}, \frac{1}{26}) \in E(G)$ and $\beta(D,f,1) < \infty$ since $D(x,y) \leq 2(x+y) \leq 4$ for any $x,y \in X$.

In addition, it is easy to prove that f is G-continuous, and g is continuous.

Next, we show that f is g-edge preserving w.r.t G, and $E(G)$ satisfies the transitivity property.

Assume $x, y, z \in X$ with $(gx, gy) \in E(G)$. We have $gx \neq 0$ or $gy = 0$ which implies $x \neq 0$ or $y = 0$. We obtain $fx \neq 0$ or $fy = 0$ so $(fx, fy) \in E(G)$ and hence f is g-edge preserving w.r.t G.

To prove transitivity property, suppose $(x, z), (z, y) \in E(G)$. Notice that $z = 0 \Rightarrow y = 0$ and $z \neq 0 \Rightarrow x \neq 0$. So $x \neq 0$ or $y = 0$ which means $(x, y) \in E(G)$.

Now, we prove that (f, g) is a BKC-contraction with $\lambda = \frac{1}{3}$. Let $x, y \in X$ such that $(gx, gy) \in E(G)$.

If $gy = 0$, then $fy = 0$ and we have

$$\begin{aligned}
D(fx, fy) &= D(\frac{x}{2x+24}, 0) \\
&= \frac{1}{3}\left(\frac{x}{2x+24} + \frac{x}{2x+24} + \frac{x}{2x+24}\right) \\
&\leq \frac{1}{3}\left(\frac{x}{16} + \frac{x}{16} + \frac{x}{2x+24}\right) \\
&= \lambda[D(gx, fy) + D(gy, fx)] \\
&\leq \lambda \max\{2D(gx, gy), D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\}.
\end{aligned}$$

If $gy \neq 0$, then $gx \neq 0$ and we have

$$\begin{aligned}
D(fx, fy) &= D(\frac{x}{2x+24}, \frac{y}{2y+24}) \\
&= 2(\frac{x}{2x+24} + \frac{y}{2y+24}) \\
&\leq 2\left(\frac{x}{24} + \frac{y}{24}\right) \\
&= \frac{2}{3}\left(\frac{x}{8} + \frac{y}{8}\right) \\
&\leq \frac{2}{3}\left(\frac{x}{8} + \frac{y}{8} + \frac{x}{8} + \frac{y}{8}\right) \\
&= \lambda[D(gx, gy) + D(gx, gy)] \\
&\leq \lambda \max\{2D(gx, gy), D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\}.
\end{aligned}$$

Finally, let us prove that f and g are D-compatible. If $\{x_n\} \subseteq X$ satisfies

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = a$$

for some $a \in X$, then

$$\lim_{n\to\infty} \frac{x_n}{8} = \lim_{n\to\infty} \frac{x_n}{2x_n + 24} = a.$$

As a consequence, we obtain $a = 0$ which gives

$$\lim_{n\to\infty} D(gfx_n, fgx_n) = 0.$$

By Theorem 2.13, there exists a coincidence point of f and g.

Example 2.18. Let $X = [0, 1]$ and let D be a generalized metric such that

$$D(x, y) = x + y.$$

Then it is not hard to see that (X, D) is D-complete.

Let us define

$$E(G) = \{(x, y) : x \geq y\}.$$

Furthermore, let $f, g : X \to X$ be defined by

$$f(x) = \ln\left(1 + \frac{x^2}{4}\right) \quad \text{and} \quad g(x) = x^2.$$

We will use Theorem 2.13 to show that $C(f, g) \neq \emptyset$.

To start with, it can be checked that $f(X) \subseteq g(X)$ and $g(X)$ is D-complete because $0 \in g(X)$. Next, we choose $x_0 = 1$ so that $(g(1), f(1)) = (1, \ln(\frac{5}{4})) \in E(G)$ and $\beta(D, f, 1) < \infty$ since $D(x, y) \leq 2(x + y) \leq 4$ for any $x, y \in X$.

In addition, it is easy to prove that f is G-continuous, and g is continuous.

Next, we show that f is g-edge preserving w.r.t G and $E(G)$ satisfies the transitivity property.

Assume $x, y, z \in X$ with $(gx, gy) \in E(G)$. We have $gx \geq gy$ which implies $x \geq y$. We obtain $fx = \ln\left(1 + \frac{x^2}{4}\right) \geq \ln\left(1 + \frac{y^2}{4}\right) = fy$, so $(fx, fy) \in E(G)$ and hence f is g-edge preserving w.r.t G.

To prove transitivity property, suppose $(x, z), (z, y) \in E(G)$. Notice that $x \geq z$ and $z \geq y$ so $x \geq y$ which means $(x, y) \in E(G)$.

Now, we prove that (f, g) is a BKC-contraction with $\lambda = \frac{1}{3}$. Let $x, y \in X$ such that $(gx, gy) \in E(G)$, we have

$$\begin{aligned}
D(fx, fy) &= D(\ln\left(1 + \frac{x^2}{4}\right), \ln\left(1 + \frac{y^2}{4}\right)) \\
&= \ln\left(1 + \frac{x^2}{4}\right) + \ln\left(1 + \frac{y^2}{4}\right) \\
&= \frac{\ln\left(1 + \frac{x^2}{4}\right)}{3} + \frac{\ln\left(1 + \frac{x^2}{4}\right)}{3} + \frac{\ln\left(1 + \frac{x^2}{4}\right)}{3} + \frac{\ln\left(1 + \frac{y^2}{4}\right)}{3} \\
&\quad + \frac{\ln\left(1 + \frac{y^2}{4}\right)}{3} + \frac{\ln\left(1 + \frac{y^2}{4}\right)}{3} \\
&\leq \frac{\frac{x^2}{2}}{3} + \frac{\frac{x^2}{2}}{3} + \frac{\ln\left(1 + \frac{x^2}{4}\right)}{3} + \frac{\frac{y^2}{2}}{3} + \frac{\frac{y^2}{2}}{3} + \frac{\ln\left(1 + \frac{y^2}{2}\right)}{3} \\
&= \lambda[D(gx, fx) + D(gy, fy)] \\
&\leq \lambda \max\{2D(gx, gy), D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\}.
\end{aligned}$$

Finally, let us prove that f and g are D-compatible. If $\{x_n\} \subseteq X$ with

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = a$$

for some $a \in X$, then $\ln\left(1 + \dfrac{a}{2}\right) = a$. As a consequence, we obtain $a = 0$ which gives

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} (x_n)^2 = 0$$

and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \ln\left(1 + \frac{(x_n)^2}{4}\right) = 0.$$

This implies

$$\lim_{n\to\infty} D(gfx_n, fgx_n) = \lim_{n\to\infty} \left(\ln\left(1+\frac{(x_n)^2}{4}\right)\right)^2 + \ln\left(1+\frac{(x_n)^4}{4}\right) = 0.$$

By Theorem 2.13, there exists a coincidence point of f and g.

3. Application

Fixed point theory has been a useful tool in the field of integral equations for a while, for example, see [12–15]. We will devote this section to an application of our results to the following integral equation.

$$x(t) = \int_0^T p(t,s,x(s))ds \tag{3.1}$$

where $t \in [0,T]$ and $T \in (0,\infty)$.

To begin with, we will assume that $X = C([0,T],\mathbb{R})$ and set

$$D(x,y) = \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |y(t)|$$

for any $x, y \in C([0,T],\mathbb{R})$. It is not hard to see that (X,D) is a D-complete generalized metric space. Furthermore, our results imply the following theorem.

Theorem 3.1. *Under the above setting, assume further that the following hold.*

(i) $p : [0,T]\times[0,T]\times\mathbb{R} \to \mathbb{R}$ *is a continuous function;*

(ii) There exists $K \in (2,\infty)$ *such that for each* $x, y \in \mathbb{R}$, *if* $x \le y$, *then* $p(t,s,x) \le p(t,s,y)$ *and*

$$|p(t,s,x)| + |p(t,s,y)| \le \frac{1}{KT}\left(|x| + |y|\right).$$

for any $s,t \in [0,T]$; *and*

(iii) There exists $x_0 \in X$ *with* $x_0(t) \ge \int_0^T p(t,s,x_0(s))ds$ *for any* $t \in [0,T]$.

Then the integral equation (3.1) *has a solution.*

Proof. First, we define $f, g : X \to X$ as follows.

$$fx(t) = \int_0^T p(t,s,x(s))ds,$$

and $gx(t) = x(t)$ for every $x \in X$, $t \in [0,T]$.

Next, we define $E(G)$ as follows.

$$E(G) = \{(x,y) : x(t) \ge y(t) \ \text{ for all } \ t \in [0,T]\}.$$

Obviously, $f(X) \subseteq g(X)$, f is G-continuous, and g is continuous.

Now, observe that assumption (iii) induces assumption (c) of Theorem 2.13 because if we start with x_0 in the assumption, then $fx_n = x_{n+1} \le x_n$ for any $n \in \mathbb{N}\cup\{0\}$. In addition, it is clear that assumption (e) of Theorem 2.13 holds in our case.

Next, we will show f is g-edge preserving w.r.t G. Notice that $(gx, gy) \in E(G)$ implies $x(t) = gx(t) \ge gy(t) = y(t)$ for any $t \in [0,T]$. Along with assumption (ii), this provides

$p(t,s,x(s)) \geq p(t,s,y(s))$ for any $s,t \in [0,T]$. As a result,

$$\begin{aligned} fx(t) &= \int_0^T p(t,s,x(s))ds \\ &\geq \int_0^T p(t,s,y(s))ds \\ &= fy(t). \end{aligned}$$

Hence, $(fx, fy) \in E(G)$ which implies that f is g-edge preserving w.r.t G.

Obviously, $E(G)$ satisfies the transitivity property.

In the last part, let us prove that (f,g) is a BKC-contraction for some $\lambda \in [0,1/2)$.

If $x(t) \geq y(t)$ for all $t \in [0,T]$, then assumption (ii) implies that for any $t \in [0,\infty)$,

$$\begin{aligned} &|fx(t)| + |fy(t)| \\ &\leq \int_0^T |p(t,s,x(s))| + |p(t,s,y(s))|ds \\ &\leq \frac{1}{KT}\int_0^T (|x(s)| + |y(s)|)ds \\ &\leq \frac{1}{K}\left(\max_{t\in[0,T]}|gx(t)| + \max_{t\in[0,T]}|gy(t)|\right) \\ &\leq \frac{1}{K}\left(2[\max_{t\in[0,T]}|gx(t)| + \max_{t\in[0,T]}|gy(t)|]\right). \end{aligned}$$

This implies that (f,g) is a BKC-contraction for $\lambda = \frac{1}{K}$.

Theorem 2.13 suggests that a coincidence point of f and g exists. By the definition of g, we have that this point is also a solution to the equation (3.1). ■

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