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HYBRID DELAY FEEDBACK CONTROL FOR MIXED H_∞ /PASSIVE SYNCHRONIZATION OF COMPLEX DYNAMICAL NETWORKS WITH TIME-VARYING AND MIXED COUPLING DELAY

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Abstract This paper considers mixed H_∞ and passive synchronization of complex dynamical networks with discrete time-varying delay for a class of mixed coupling and hybrid delay feedback control. The discrete delay is assumed to be interval time-varying delay, which means that the upper and lower bound of delay are adopted. The purpose of this work provided by designing hybrid delay feedback control, which contains error linear term and time-varying delay error linear term, using appropriate Lyapunov-Krasovkii functional (LKF) deal with some new integral inequalities and combined with improved reciprocally convex such that the synchronization error system is exponentially stable with a mixed H_∞ and passive performance index simultaneously. The sufficient conditions are presented in the term of linear matrix inequalities (LMIs). Finally, the effectiveness of our proposed method is employed to demonstrate via numerical example.

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1. Introduction

Over the past few decades, complex dynamical networks (CDNs) have been widely studied by abundant researchers. It is structure of a large set of interconnected nodes in which each node is a unit with their respective dynamical equations. In recent years, complex networks have been an attractive research topic since it can be used to model

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and describe many phenomena in nature such as the World Wide Web, cellular and metabolic networks, transportation networks, communication networks, social networks, and electrical power grids etc.

Synchronization of CDNs is one of the most important and interesting subject in a lot of research and application fields. Many phenomena in nature are described by the synchronization of complex networks, which have been published in [5, 16, 18]. Moreover, a lot of synchronization criteria for CDNs have been proposed. For instant, [20] introduced simple uniform networks model without time delays and they can achieve synchronization of small-world dynamical network. However, in reality time delay is an unavoidable situation in real dynamic systems. It is well known that there exists in a network may lead to instability, poor performances, oscillations and divergences. There are many examples that can be found in networks such as application engineering, electrical power networks, physical networks and so forth. CDNs, especially are caused by the fitness of signal transmission and switching speed [3]. So, Li and Chen, [6] investigated a complex dynamical network model with a coupling delay for both continuous-time and discrete-time cases and they can derive synchronization conditions for both delay-independent and delay-dependent asymptotical stabilities. Gao et al. [1] studied synchronization for general of complex dynamical networks with coupling delays which they can find new criterion for guaranteeing the asymptotically stable of synchronized states. As we all know, time-varying delays case which is more regular than the constant time delay. So, there are many of researchers challenge this topic which their works available in literature [7, 14, 26]. In the past decades, synchronization control problems have been received considerable attention. These issue occurred for achieving synchronization for an asynchronous complex network for example, impulsive control [25], feedback control [8], pinning control [22], sample-data control [9, 21] and so forth.

On the other hand, the passivity theory acts as vital part in analysis of linear and nonlinear systems, which has background knowledge in circuit theory [4]. It has been a hot subject in analysis and design of networks control which has been available in [12]. The concept of passive synchronization control is to remain the system internally stable. Furthermore, H_∞ synchronization control is important tool for stable effectiveness of disturbances or noises in CDNs [2, 17]. In control theory, H_∞ approach plays an important role in the synthesizing controllers to achieve stabilization with guaranteed performance. In recently year, the problem of mixed H_∞ and passive has been become a popular topic in various research. It was first presented in [10, 11]. The coping with the mixed H_∞ and passive analysis is proposed. For example, Shen et al. [19] investigated the problem of mixed H_∞ and passive synchronization of complex networks with time-varying delay via a sampled-data control and they derived the sufficient conditions which lead to stable system and satisfy mixed H_∞ and passive performance level. Wang et al. [23] studied on the topic of mixed H_∞ and passive sampled-data synchronization control of CDNs which pay attention to distributed coupling delay.

Inspired by the above discussions, in this paper, we aim to design a set of hybrid delay feedback controllers which composed of error linear term and time-varying delay error linear term to achieve mixed H_∞ and passive synchronization of complex dynamical networks with time-varying delay and mixed coupling. In our works, we focus on interval time-varying delay which some researchers have not studied yet. By constructing an appropriate Lyapunov-Krasovskii functional comprising double and triple integral terms and using the information on the boundedness of the time-varying delay, improved Jensen inequalities together with reciprocally convex combination which the sufficient conditions are derived for analysis of the problem. Finally, illustrative example is provided to show the effectiveness of the proposed methods.

2. Preliminaries

Consider the following complex dynamical network with N identical coupled nodes

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + g(x_i(t-h(t))) + c_1 \sum_{j=1}^{N} a_{ij} D_1 x_j(t) + c_2 \sum_{j=1}^{N} b_{ij} D_2 x_j(t-h(t)) \\ &\quad + u_i(t) + \omega_i(t), \qquad (2.1) \\ y_i(t) &= \tilde{Z} x_i(t), \quad i = 1, 2, \ldots, N, \qquad (2.2) \end{aligned}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ are respectively, the state vector and the control input of the node i. $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous nonlinear vector valued functions. The function $h(t)$ denotes the time-varying delay satisfying:

$$0 \leq \lambda_1 \leq h(t) \leq \lambda_2, \quad 0 \leq \dot{h}(t) \leq \gamma < 1, \qquad (2.3)$$

where λ_1, λ_2 and γ are known positive constants. The constant $c_1 > 0$ and $c_2 > 0$ denote the non-delayed and delayed coupling strength, respectively, D_1 and D_2 are constant inner-coupling matrices, $\omega_i(t) \in \mathbb{R}^n$ is the external disturbance which belongs to $\mathcal{L}_2[0,\infty), \tilde{Z}$ is a known matrix with appropriate dimensions. $A = (a_{ij})_{N\times N}, B = (b_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ are the outer-coupling matrices of the non delay and the time-varying delay of the network in which a_{ij}, b_{ij} are defined as follows: if there is a connection between node i and node $j (j \neq i)$, then $a_{ij} = a_{ji} = 1, b_{ij} = b_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0, b_{ij} = b_{ji} = 0$, and the diagonal elements of matrices A and B are defined by

$$a_{ii} = -\sum_{j=1, i\neq j}^{N} a_{ij}, \qquad b_{ii} = -\sum_{j=1, i\neq j}^{N} b_{ij}, \qquad i = 1, 2, \ldots, N. \qquad (2.4)$$

The system of the unforced isolate node is given as follows

$$\dot{s}(t) = f(s(t)) + g(s(t-h(t))), \qquad (2.5)$$

$$y_a(t) = \tilde{Z} s(t), \qquad (2.6)$$

$y_a(t)$ is the unforced isolate output, we define the synchronization error as $e_i(t) = x_i(t) - s(t)$. Then, the error dynamics of complex network (2.1) and (2.2) can be transformed as follows:

$$\begin{aligned} \dot{e}_i(t) &= \xi(e_i(t)) + \zeta(e_i(t-h(t))) + c_1 \sum_{j=1}^{N} a_{ij} D_1 e_j(t) + c_2 \sum_{j=1}^{N} b_{ij} D_2 e_j(t-h(t)) \\ &\quad + u_i(t) + \omega_i(t), \qquad (2.7) \\ \tilde{y}_i(t) &= \tilde{Z} e_i(t), \quad i = 1, 2, \ldots, N, \qquad (2.8) \end{aligned}$$

where $\xi(e_i(t)) = f(x_i(t)) - f(s(t)),\ \tilde{y}_i(t) = y_i(t) - y_a(t)$
$\zeta(e_i(t-h(t))) = g(x_i(t-h(t))) - g(s(t-h(t)))$.

In this work, we design hybrid feedback control as follows:

$$u_i(t) = K_i e_i(t) + \overline{K}_i e_i(t - d(t)), \qquad i = 1, 2, 3, \dots, N, \tag{2.9}$$

where K_i and $\overline{K}_i$ are the feedback gain to be determined and the time-varying delay function $d(t)$ satisfies the condition

$$0 \le d(t) \le d, \qquad 0 \le \dot{d}(t) \le \beta < 1, \tag{2.10}$$

where d and β are known constants. For given matrices R_1, R_2, S_1 and S_2 the nonlinear function f, g satisfy

$$\begin{aligned} &[f(x) - f(y) - R_1(x - y)]^T [f(x) - f(y) - S_1(x - y)] \le 0, \\ &[g(x) - g(y) - R_2(x - y)]^T [g(x) - g(y) - S_2(x - y)] \le 0, \ \forall x, y \in \mathbb{R}^n. \end{aligned} \tag{2.11}$$

By substituting (2.9) into (2.7), such that the resulting closed-loop error system

$$\begin{aligned} \dot{e}_i(t) &= \xi(e_i(t)) + \zeta(e_i(t - h(t))) + c_1 \sum_{j=1}^{N} a_{ij} D_1 e_j(t) + c_2 \sum_{j=1}^{N} b_{ij} D_2 e_j(t - h(t)) \\ &\quad + K_i e_i(t) + \overline{K}_i e_i(t - d(t)) + \omega_i(t), \qquad (2.12) \\ \tilde{y}_i(t) &= \tilde{Z} e_i(t) \qquad i = 1, 2, \dots, N. \qquad (2.13) \end{aligned}$$

The equation (2.12) and (2.13) can be simplified as the following form:

$$\begin{aligned} \dot{e}(t) &= \tilde{\xi}(e(t)) + \tilde{\zeta}(e(t - h(t))) + c_1 (A \otimes D_1) e(t) + c_2 (B \otimes D_2) e(t - h(t)) \\ &\quad + K e(t) + \overline{K} e(t - d(t)) + \omega(t), \qquad (2.14) \\ y(t) &= Z e(t), \qquad (2.15) \end{aligned}$$

where $e(\cdot) = [e_1^T(\cdot)\ e_2^T(\cdot) \dots e_N^T(\cdot)]^T, K = \text{diag}\{K_1, K_2, \dots, K_N\}$,
$\overline{K} = \text{diag}\{\overline{K}_1, \overline{K}_2, \dots, \overline{K}_N\}, y(t) = [y_1^T(t)\ y_2^T(t) \dots y_N^T(t)]^T$,
$Z = \text{diag}\left\{\tilde{Z}, \tilde{Z}, \dots, \tilde{Z}\right\}, \tilde{\xi}(e(t)) = [\xi^T(e_1(t))\ \xi^T(e_2(t)) \dots \xi^T(e_N(t))]^T$,
$\tilde{\zeta}(e(t - h(t))) = [\zeta^T(e_1(t - h(t)))\ \zeta^T(e_2(t - h(t))) \dots \zeta^T(e_N(t - h(t)))]^T$,
$\omega(t) = [\omega_1^T(t)\ \omega_2^T(t) \dots \omega_N^T(t)]^T$.

Remark 2.1. If $g(\cdot) = 0$, $c_1 = 1$, $c_2 = 1$, $a_{ij} = b_{ij}$, the network model (2.1) turns into the complex dynamical network proposed by [19]

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + \sum_{j=1}^{N} E_{ij} B x_j(t) + \sum_{j=1}^{N} E_{ij} A x_j(t - \tau(t)) + u_i(t) \\ &\quad + \omega_i(t) \end{aligned} \tag{2.16}$$

If $g(\cdot) = 0, c_1 = c_2 = 1, a_{ij} = b_{ij}, \omega_i(t) = 0$ the network model (2.1) turns into the complex dynamical network presented by [21]

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} G_{ij} D x_j(t) + \sum_{j=1}^{N} G_{ij} A x_j(t - \tau(t)) + u_i(t) \tag{2.17}$$

Hence, our model (2.1) is general networks model, with (2.16), 2.17) as the special case.

The following definition and lemmas are introduced for deriving the main result.

Definition 2.2. [19] The system (2.14) and (2.15) with $\omega(t) = 0$ are exponentially stable, if there exist two constants $\eta > 0$ and $\varpi > 0$ such that

$$\|e(t)\|^2 \leq \eta e^{-\varpi t} \max\{\sup_{-\max\{\lambda_2,d\}\leq\theta\leq 0} \|e(\theta)\|^2, \sup_{-\max\{\lambda_2,d\}\leq\theta\leq 0} \|\dot{e}(\theta)\|^2\}.$$

Definition 2.3. [19] For given scalar $\sigma \in [0,1]$, the system (2.14) and (2.15) are exponentially stable with a mixed H_∞ and passivity performance index δ, if the following two conditions can be guaranteed simultaneously:

(1) the system (2.14) and (2.15) are exponentially stable in view of Definition 2.2.
(2) under zero original condition, there exists a scalar $\delta > 0$ such that the following inequality is satisfied:

$$\int_0^{\mathcal{T}_p} \left[-\sigma z^T(t)z(t) + 2(1-\sigma)\delta z^T(t)\omega(t)\right] dt \geq -\delta^2 \int_0^{\mathcal{T}_p} \left[\omega^T(t)\omega(t)\right] dt, \tag{2.18}$$

for any $\mathcal{T}_p \geq 0$ and any non-zero $\omega(t) \in \mathcal{L}_2[0,\infty)$.

Lemma 2.4. *[24] For a positive definite matrix $R > 0$, and any continuously differentiable function $x : [a,b] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{b-a}\omega_1 R\omega_1 + \frac{3}{b-a}\omega_2^T R\omega_2 + \frac{5}{b-a}\omega_3^T R\omega_3, \tag{2.19}$$

where

$$\begin{aligned} \omega_1 &= x(b) - x(a), \\ \omega_2 &= x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds, \\ \omega_3 &= x(b) - x(a) + \frac{6}{b-a}\int_a^b x(s)ds - \frac{12}{(b-a)^2}\int_a^b \int_u^b x(s)dsdu. \end{aligned}$$

Lemma 2.5. *(Jensen's inequality [15]) Suppose $x(t) \in \mathbb{R}^n$, for any positive definite matrix P the following inequality holds:*

$$\int_a^b x^T(s)Px(s)\,ds \geq \frac{1}{b-a}\int_a^b x^T(s)\,dsP\int_a^b x(s)\,ds. \tag{2.20}$$

Lemma 2.6. *[24] For a positive definite matrix $R > 0$, and any continuously differentiable function $x : [a,b] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_a^b \int_u^b \dot{x}^T(s)R\dot{x}(s)dsdu \geq 2\omega_4^T R\omega_4 + 4\omega_5^T R\omega_5 + 6\omega_6^T R\omega_6, \tag{2.21}$$

where

$$
\begin{aligned}
\omega_4 &= x(b) - \frac{1}{b-a}\int_a^b x(s)ds, \\
\omega_5 &= x(b) + \frac{2}{b-a}\int_a^b x(s)ds - \frac{6}{(b-a)^2}\int_a^b\int_u^b x(s)dsdu, \\
\omega_6 &= x(b) - \frac{3}{b-a}\int_a^b x(s)ds + \frac{24}{(b-a)^2}\int_a^b\int_u^b x(s)dsdu \\
&\quad -\frac{60}{(b-a)^3}\int_a^b\int_u^b\int_s^b x(r)drdsdu.
\end{aligned}
$$

Lemma 2.7. *(Reciprocal convexity lemma [13]) For any vectors x_1, x_2, matrices $U > 0, V$, and real scalars $\alpha \geq 0$, $\beta \geq 0$ satisfying $\alpha + \beta = 1$, the following inequality holds:*

$$
-\frac{1}{\alpha}x_1^TUx_1 - \frac{1}{\beta}x_2^TUx_2 \leq -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} U & V \\ V^T & U \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.22}
$$

subject to

$$
0 < \begin{bmatrix} U & V \\ V^T & U \end{bmatrix}.
$$

3. Main Results

Firstly, we present a sufficient condition which guarantees the synchronizaion error system (2.14) and (2.15) to be exponentially stable and satisfy with a prescribed mixed H_∞ and passivity performance level δ. Then the desired controllers are proposed. For simplicity, we denote

$$
\begin{aligned}
\Pi_1 &= r_1 - r_6, \\
\Pi_2 &= [r_1^T - r_4^T, r_1^T + r_4^T - 2r_8^T, r_1^T - r_4^T + 6r_8^T - 12r_{10}^T, r_4^T - r_5^T, r_4^T + r_5^T - 2r_9^T, \\
&\quad r_4^T - r_5^T + 6r_9^T - 12r_{11}^T]^T, \\
\Pi_3 &= \left[r_1^T - r_3^T, r_1^T + r_3^T - 2r_{12}^T, r_1^T - r_3^T + 6r_{12}^T - 12r_{14}^T, r_3^T - r_6^T, r_3^T + r_6^T - 2r_{13}^T, \right. \\
&\quad \left. r_3^T - r_6^T + 6r_{13}^T - 12r_{15}^T\right]^T, \\
\Pi_4 &= r_1 - r_3, \\
\Pi_5 &= \left[r_1^T - r_{12}^T, r_1^T + 2r_{12}^T - 6r_{14}^T, r_1^T - 3r_{12}^T + 24r_{14}^T - 60r_{17}^T\right]^T, \qquad (3.1) \\
\Pi_6 &= \left[r_1^T - r_8^T, r_1^T + 2r_8^T - 6r_{10}^T, r_1^T - 3r_8^T + 24r_{10}^T - 60r_{16}^T\right]^T, \\
\Pi_7 &= \left[r_1^T, r_{18}^T\right]^T, \\
\Pi_8 &= \left[r_3^T, r_{19}^T\right]^T, \\
\overline{R}_{\Lambda_i} &= \frac{R_{\Lambda_i}^T S_{\Lambda_i} + S_{\Lambda_i}^T R_{\Lambda_i}}{2}, \quad \overline{S}_{\Lambda_i} = -\frac{R_{\Lambda_i}^T + S_{\Lambda_i}^T}{2}, \\
R_{\Lambda_i} &= \mathrm{diag}_N\{R_i\}, \quad S_{\Lambda_i} = \mathrm{diag}_N\{S_i\}, \ i = 1, 2 \\
\Phi_1 &= \begin{bmatrix} \mathrm{diag}\{U_2, 3U_2, 5U_2\} & X \\ * & \mathrm{diag}\{U_2, 3U_2, 5U_2\} \end{bmatrix}, \\
\Phi_2 &= \begin{bmatrix} \mathrm{diag}\{U_3, 3U_3, 5U_3\} & Y \\ * & \mathrm{diag}\{U_3, 3U_3, 5U_3\} \end{bmatrix}, \\
\Theta_1 &= \begin{bmatrix} 2G_1 & 0 & 0 \\ 0 & 4G_1 & 0 \\ 0 & 0 & 6G_1 \end{bmatrix}, \\
\Theta_2 &= \begin{bmatrix} 2G_2 & 0 & 0 \\ 0 & 4G_2 & 0 \\ 0 & 0 & 6G_2 \end{bmatrix}, \qquad (3.2)
\end{aligned}
$$

$r_i \in \mathbb{R}^{nN \times 20nN}$ is defined as $r_i = \left[0_{nN \times (i-1)nN}, \ \ I_{nN}, \ \ 0_{nN \times (20-i)nN}\right]$
for $i = 1, 2, \ldots, 20$.

$$\begin{aligned}\chi(t) &= \big[e^T(t),\ \dot{e}^T(t),\ e^T(t-h(t)),\ e^T(t-d(t)), e^T(t-d),\ e^T(t-\lambda_1), \\ &\quad e^T(t-\lambda_2),\ \frac{1}{d(t)}\int_{t-d(t)}^{t} e^T(s)ds,\ \frac{1}{d-d(t)}\int_{t-d}^{t-d(t)} e^T(s)ds, \\ &\quad \frac{1}{d^2(t)}\int_{t-d(t)}^{t}\int_{u}^{t} e^T(s)dsdu,\ \frac{1}{(d-d(t))^2}\int_{t-d}^{t-d(t)}\int_{u}^{t-d(t)} e^T(s)dsdu, \\ &\quad \frac{1}{h(t)}\int_{t-h(t)}^{t} e^T(s)ds,\ \frac{1}{\lambda_2-h(t)}\int_{t-\lambda_2}^{t-h(t)} e^T(s)dsdu, \\ &\quad \frac{1}{h^2(t)}\int_{t-h(t)}^{t}\int_{u}^{t} e^T(s)dsdu,\ \frac{1}{(\lambda_2-h(t))^2}\int_{t-\lambda_2}^{t-h(t)}\int_{u}^{t-h(t)} e^T(s)dsdu, \\ &\quad \frac{1}{d^3(t)}\int_{t-d(t)}^{t}\int_{u}^{t}\int_{s}^{t} e^T(r)drdsdu,\ \frac{1}{h^3(t)}\int_{t-h(t)}^{t}\int_{u}^{t}\int_{s}^{t} e^T(r)drdsdu, \\ &\quad \tilde{\xi}^T(e(t)),\ \tilde{\zeta}^T(e(t-h(t)),\ \omega(t)\big]^T. \end{aligned} \tag{3.3}$$

Theorem 3.1. *For given scalars $\lambda_1, \lambda_2, \gamma, \beta, d, \varepsilon_1$ and ε_2, matrices $R_1, R_2, S_1,$ and S_2, if there exist matrices $P = \operatorname{diag}\{P_1, P_2, \ldots, P_N\} > 0, T_i > 0,\ U_i > 0\ (i = 1, 2, 3, 4), G_1 > 0, G_2 > 0, \Upsilon_1, \Upsilon_2$*
$X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{bmatrix} \in \mathbb{R}^{3nN\times 3nN}, Y = \begin{bmatrix} Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & Y_6 \\ Y_7 & Y_8 & Y_9 \end{bmatrix} \in \mathbb{R}^{3nN\times 3nN}$ *such that:*

$$\begin{aligned}\Psi &= 2r_1^TPr_2 + r_1^TT_1r_1 - r_6^TT_1r_6 + r_1^TT_2r_1 - r_7^TT_2r_7 + r_1^TT_3r_1 - (1-\gamma)\, r_3^TT_3r_3 \\ &\quad + r_1^TT_4r_1 - r_5^TT_4r_5 + \lambda_1^2r_2^TU_1r_2 - \Pi_1^TU_1\Pi_1 + d^2r_2^TU_2r_2 - \Pi_2^TU_2\Pi_1\Pi_2 \\ &\quad + \lambda_2^2r_2^TU_3r_2 - \Pi_3^T\Phi_2\Pi_3 + \lambda_2^2r_2^TU_4r_2 - (1-\gamma)\,\Pi_4^TU_4\Pi_4 + \frac{\lambda_2^2}{2}r_2^TG_1r_2 \\ &\quad - (1-\gamma)\Pi_5^T\Theta_1\Pi_5 + \frac{d^2}{2}r_2^TG_2r_2 - (1-\beta)\Pi_6^T\Theta_2\Pi_6 - 2r_1^T\Upsilon_1^Tr_2 + 2r_1^T\Upsilon_1^Tr_{18} \\ &\quad + 2r_1^T\Upsilon_1^Tr_{19} + 2r_1^T\Upsilon_1^Tc_1\,(A\otimes D_1)\,r_1 + 2r_1^T\Upsilon_1^Tc_2\,(B\otimes D_2)\,r_3 + 2r_1^T\Upsilon_1^TKr_1 \\ &\quad + 2r_1^T\Upsilon_1^T\overline{K}r_4 - 2r_2^T\Upsilon_2^Tr_2 + 2r_2^T\Upsilon_2^Tr_{18} + 2r_2^T\Upsilon_2^Tr_{19} + 2r_2^T\Upsilon_2^Tc_1\,(A\otimes D_1)\,r_1 \\ &\quad + 2r_2^T\Upsilon_2^Tc_2\,(B\otimes D_2)\,r_3 + 2r_2^T\Upsilon_2^TKr_1 + 2r_2^T\Upsilon_2^T\overline{K}r_4 + 2r_1^T\Upsilon_1^Tr_{20} \\ &\quad + 2r_2^T\Upsilon_2^Tr_{20} - \varepsilon_1\Pi_7^T\begin{bmatrix}\overline{R}_{\Lambda_1} & \overline{S}_{\Lambda_1} \\ * & I\end{bmatrix}\Pi_7 - \varepsilon_2\Pi_8^T\begin{bmatrix}\overline{R}_{\Lambda_2} & \overline{S}_{\Lambda_2} \\ * & I\end{bmatrix}\Pi_8 + r_1^T\sigma Z^TZr_1 \\ &\quad - 2(1-\sigma)\delta r_1^TZ^Tr_{20} - r_{20}^T\delta^2Ir_{20} < 0, \end{aligned} \tag{3.4}$$

$$\Phi_1 \geq 0, \tag{3.5}$$

$$\Phi_2 \geq 0, \tag{3.6}$$

then, the synchronization error system (2.14) and (2.15) are exponentially stable with a prescribed mixed H_∞ and passivity performance level δ.

Proof. Consider a LyapunovKrasovskii functional candidate for system (2.14) and (2.15)

$$V(t) = \sum_{i=1}^{11} V_i(t),$$

where

$$\begin{aligned} V_1(t) &= e^T(t)Pe(t) \\ V_2(t) &= \int_{t-\lambda_1}^{t} e^T(s)T_1e(s)ds \\ V_3(t) &= \int_{t-\lambda_2}^{t} e^T(s)T_2e(s)ds \\ V_4(t) &= \int_{t-h(t)}^{t} e^T(s)T_3e(s)ds \\ V_5(t) &= \int_{t-d}^{t} e^T(s)T_4e(s)ds \\ V_6(t) &= \lambda_1 \int_{-\lambda_1}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)U_1\dot{e}(s)dsd\theta \\ V_7(t) &= d \int_{-d}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)U_2\dot{e}(s)dsd\theta \\ V_8(t) &= \lambda_2 \int_{-\lambda_2}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)U_3\dot{e}(s)dsd\theta \\ V_9(t) &= \lambda_2 \int_{-h(t)}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)U_4\dot{e}(s)dsd\theta \\ V_{10}(t) &= \int_{t-h(t)}^{t}\int_{\tau}^{t}\int_{\theta}^{t} \dot{e}^T(s)G_1\dot{e}(s)dsd\theta d\tau \\ V_{11}(t) &= \int_{t-d(t)}^{t}\int_{\tau}^{t}\int_{\theta}^{t} \dot{e}^T(s)G_2\dot{e}(s)dsd\theta d\tau. \end{aligned} \tag{3.7}$$

Calculating the time derivative of $V(t)$ along the trajectories of system (2.14) and (2.15) give the following result:

$$\dot{V}_1(t) = 2e^T(t)P\dot{e}(t), \tag{3.8}$$

$$\dot{V}_2(t) = e^T(t)T_1e(t) - e^T(t-\lambda_1)T_1e(t-\lambda_1), \tag{3.9}$$

$$\dot{V}_3(t) = e^T(t)T_2e(t) - e^T(t-\lambda_2)T_2e(t-\lambda_2), \tag{3.10}$$

$$\begin{aligned} \dot{V}_4(t) &= e^T(t)T_3e(t) - (1-\dot{h}(t))e^T(t-h(t))T_3e(t-h(t)) \\ &\leq e^T(t)T_3e(t) - (1-\gamma)e^T(t-h(t))T_3e(t-h(t)), \end{aligned} \tag{3.11}$$

$$\dot{V}_5(t) = e^T(t)T_4e(t) - e^T(t-d)T_4e(t-d), \tag{3.12}$$

$$\dot{V}_6(t) = \lambda_1^2\dot{e}^T(t)U_1\dot{e}(t) - \lambda_1\int_{t-\lambda_1}^{t} \dot{e}^T(s)U_1\dot{e}(s)ds, \tag{3.13}$$

$$\dot{V}_7(t) = d^2\dot{e}^T(t)U_2\dot{e}(t) - d\int_{t-d}^{t} \dot{e}^T(s)U_2\dot{e}(s)ds, \tag{3.14}$$

$$\dot{V}_8(t) = \lambda_2^2\dot{e}^T(t)U_3\dot{e}(t) - \lambda_2\int_{t-\lambda_2}^{t} \dot{e}^T(s)U_3\dot{e}(s)ds, \tag{3.15}$$

$$\begin{aligned}
\dot{V}_9(t) &= \lambda_2 h(t)\dot{e}^T(t)U_4\dot{e}(t) - \lambda_2(1-\dot{h}(t))\int_{t-h(t)}^{t}\dot{e}^T(s)U_4\dot{e}(s)ds \\
&\leq \lambda_2^2\dot{e}^T(t)U_4\dot{e}(t) - \lambda_2(1-\dot{h}(t))\int_{t-h(t)}^{t}\dot{e}^T(s)U_4\dot{e}(s)ds, \qquad (3.16)\\
\dot{V}_{10}(t) &= \frac{h^2(t)}{2}\dot{e}^T(t)G_1\dot{e}(t) - (1-\dot{h}(t))\int_{t-h(t)}^{t}\int_{s}^{t}\dot{e}^T(s)G_1\dot{e}(s)dsd\theta \\
&\leq \frac{\lambda_2^2}{2}\dot{e}^T(t)G_1\dot{e}(t) - (1-\gamma)\int_{t-h(t)}^{t}\int_{s}^{t}\dot{e}^T(s)G_1\dot{e}(s)dsd\theta, \\
\dot{V}_{11}(t) &= \frac{d^2(t)}{2}\dot{e}^T(t)G_2\dot{e}(t) - (1-\dot{d}(t))\int_{t-d(t)}^{t}\int_{s}^{t}\dot{e}^T(s)G_2\dot{e}(s)dsd\theta \\
&\leq \frac{d^2}{2}\dot{e}^T(t)G_2\dot{e}(t) - (1-\beta)\int_{t-d(t)}^{t}\int_{s}^{t}\dot{e}^T(s)G_2\dot{e}(s)dsd\theta. \qquad (3.17)
\end{aligned}$$

Utilizing Lemma 2.5, the following relation is easily obtained

$$\begin{aligned}
-\lambda_1\int_{t-\lambda_1}^{t}\dot{e}^T(s)U_1\dot{e}(s)ds &\leq -\left(\int_{t-\lambda_1}^{t}\dot{e}^T(s)ds\right)^T U_1\left(\int_{t-\lambda_1}^{t}\dot{e}(s)ds\right) \\
&= -\chi^T(t)\Pi_1^T U_1\Pi_1\chi(t). \qquad (3.18)
\end{aligned}$$

Apply with Lemma 2.4 and the reciprocally convex approach, we have

$$\begin{aligned}
& -d\int_{t-d}^{t}\dot{e}^T(s)U_2\dot{e}(s)ds \\
= & -d\int_{t-d}^{t-d(t)}\dot{e}^T(s)U_2\dot{e}(s)ds - d\int_{t-d(t)}^{t}\dot{e}^T(s)U_2\dot{e}(s)ds \\
\leq & -\frac{d}{d-d(t)}\left[e(t-d(t))-e(t-d)\right]^T U_3\left[e(t-d(t))-e(t-d)\right] \\
& -\frac{d}{d-d(t)}\left[e(t-d(t))+e(t-d)-\frac{2}{d-d(t)}\int_{t-d}^{t-d(t)}e(s)ds\right]^T \\
& \times 3U_2\left[e(t-d(t))+e(t-d)-\frac{2}{d-d(t)}\int_{t-d}^{t-d(t)}e(s)ds\right] \\
& -\frac{d}{d-d(t)}\left[e(t-d(t))+e(t-d)+\frac{6}{d-d(t)}\int_{t-d}^{t-d(t)}e(s)ds\right. \\
& \left.-\frac{12}{(d-d(t))^2}\int_{t-d}^{t-d(t)}\int_{u}^{t-d(t)}e(s)dsdu\right]^T 5U_2\left[e(t-d(t))+e(t-d)\right. \\
& \left.+\frac{6}{d-d(t)}\int_{t-d}^{t-d(t)}e(s)ds-\frac{12}{(d-d(t))^2}\int_{t-d}^{t-d(t)}\int_{u}^{t-d(t)}e(s)dsdu\right]
\end{aligned}$$

$$-\frac{d}{d(t)}\left[e(t)-e(t-d(t))\right]^T U_2\left[e(t)-e(t-d(t))\right] \tag{3.19}$$

$$-\frac{d}{d(t)}\left[e(t)+e(t-d(t))-\frac{2}{d(t)}\int_{t-d(t)}^{t}e(s)ds\right]^T 3U_2\left[e(t)+e(t-d(t))\right.$$

$$\left.-\frac{2}{d(t)}\int_{t-d(t)}^{t}e(s)ds\right]$$

$$\frac{-d}{d(t)}\left[e(t)-e(t-d(t))+\frac{6}{d(t)}\int_{t-d(t)}^{t}e(s)ds-\frac{12}{d^2(t)}\int_{t-d(t)}^{t}\int_{u}^{t}e(s)dsdu\right]^T$$

$$\times 5U_2\left[e(t)-e(t-d(t))+\frac{6}{d(t)}\int_{t-d(t)}^{t}e(s)ds-\frac{12}{d^2(t)}\int_{t-d(t)}^{t}\int_{u}^{t}e(s)dsdu\right]$$

$$-\frac{d}{d(t)}\begin{bmatrix} e(t)-e(t-d(t)) \\ e(t)+e(t-d(t))-\frac{2}{d(t)}\int_{t-d(t)}^{t}e(s)ds \\ e(t)-e(t-d(t))+\frac{6}{d(t)}\int_{t-d(t)}^{t}e(s)ds-\frac{12}{d^2(t)}\int_{t-d(t)}^{t}\int_{u}^{t}e(s)dsdu \end{bmatrix}^T$$

$$\times\begin{bmatrix} U_2 & 0 & 0 \\ 0 & 3U_2 & 0 \\ 0 & 0 & 5U_2 \end{bmatrix}\times$$

$$\begin{bmatrix} e(t)-e(t-d(t)) \\ e(t)+e(t-d(t))-\frac{2}{d(t)}\int_{t-d(t)}^{t}e(s)ds \\ e(t)-e(t-d(t))+\frac{6}{d(t)}\int_{t-d(t)}^{t}e(s)ds-\frac{12}{d^2(t)}\int_{t-d(t)}^{t}\int_{u}^{t}e(s)dsdu \end{bmatrix}$$

$$-\frac{d}{d-d(t)}\times$$

$$\begin{bmatrix} e(t-d(t))-e(t-d) \\ e(t-d(t))+e(t-d)-\frac{2}{d-d(t)}\int_{t-d}^{t-d(t)}e(s)ds \\ e(t-d(t))-e(t-d)+\frac{6}{d-d(t)}\int_{t-d}^{t-d(t)}e(s)ds-\frac{12}{(d-d(t))^2}\int_{t-d}^{t-d(t)}\int_{u}^{t-d(t)}e(s)dsdu \end{bmatrix}^T$$

$$\times\begin{bmatrix} U_2 & 0 & 0 \\ 0 & 3U_2 & 0 \\ 0 & 0 & 5U_2 \end{bmatrix}\times$$

$$\begin{bmatrix} e(t-d(t))-e(t-d) \\ e(t-d(t))+e(t-d)-\frac{2}{d-d(t)}\int_{t-d}^{t-d(t)}e(s)ds \\ e(t-d(t))-e(t-d)+\frac{6}{d-d(t)}\int_{t-d}^{t-d(t)}e(s)ds-\frac{12}{(d-d(t))^2}\int_{t-d}^{t-d(t)}\int_{u}^{t-d(t)}e(s)dsdu \end{bmatrix}$$

$$\leq -\chi^T(t)\Pi_2^T\Phi_1\Pi_2\chi(t). \tag{3.20}$$

Utilizing Lemma 2.4 and the reciprocally convex approach, we get

$$\begin{aligned} & -\lambda_2\int_{t-\lambda_2}^{t}\dot{e}^T(s)U_3\dot{e}(s)ds \\ = & -\lambda_2\int_{t-\lambda_2}^{t-h(t)}\dot{e}^T(s)U_3\dot{e}(s)ds-\lambda_2\int_{t-h(t)}^{t}\dot{e}^T(s)U_3\dot{e}(s)ds \\ \leq & -\frac{\lambda_2}{\lambda_2-h(t)}\left[e(t-h(t))-e(t-\lambda_2)\right]^T U_3\left[e(t-h(t))-e(t-\lambda_2)\right] \end{aligned}$$

$$-\frac{\lambda_2}{\lambda_2-h(t)}\left[e(t-h(t))+e(t-\lambda_2)-\frac{2}{\lambda_2-h(t)}\int_{t-\lambda_2}^{t-h(t)}e(s)ds\right]^T$$
$$\times 3U_3\left[e(t-h(t))+e(t-\lambda_2)-\frac{2}{\lambda_2-h(t)}\int_{t-\lambda_2}^{t-h(t)}e(s)ds\right]$$
$$-\frac{\lambda_2}{\lambda_2-h(t)}\left[e(t-h(t))+e(t-\lambda_2)+\frac{6}{\lambda_2-h(t)}\int_{t-\lambda_2}^{t-h(t)}e(s)ds\right.$$
$$\left.-\frac{12}{(\lambda_2-h(t))^2}\int_{t-\lambda_2}^{t-h(t)}\int_{u}^{t-h(t)}e(s)dsdu\right]^T 5U_3$$
$$\times\left[e(t-h(t))+e(t-\lambda_2)+\frac{6}{\lambda_2-h(t)}\int_{t-\lambda_2}^{t-h(t)}e(s)ds-\right.$$
$$\left.\frac{12}{(\lambda_2-h(t))^2}\int_{t-\lambda_2}^{t-h(t)}\int_{u}^{t-h(t)}e(s)dsdu\right]$$
$$-\frac{\lambda_2}{h(t)}\left[e(t)-e(t-h(t))\right]^T U_3\left[e(t)-e(t-h(t))\right]$$
$$-\frac{\lambda_2}{h(t)}\left[e(t)+e(t-h(t))-\frac{2}{h(t)}\int_{t-h(t)}^{t}e(s)ds\right]^T 3U_3$$
$$\times\left[e(t)+e(t-h(t))-\frac{2}{h(t)}\int_{t-h(t)}^{t}e(s)ds\right]$$
$$\frac{-\lambda_2}{h(t)}\left[e(t)-e(t-h(t))+\frac{6}{h(t)}\int_{t-h(t)}^{t}e(s)ds-\frac{12}{h^2(t)}\int_{t-h(t)}^{t}\int_{u}^{t}e(s)dsdu\right]^T$$
$$\times 5U_3\left[e(t)-e(t-h(t))+\frac{6}{h(t)}\int_{t-h(t)}^{t}e(s)ds-\frac{12}{h^2(t)}\int_{t-h(t)}^{t}\int_{u}^{t}e(s)dsdu\right]$$
$$-\frac{\lambda_2}{h(t)}\begin{bmatrix} e(t)-e(t-h(t)) \\ e(t)+e(t-h(t))-\frac{2}{h(t)}\int_{t-h(t)}^{t}e(s)ds \\ e(t)-e(t-h(t))+\frac{6}{h(t)}\int_{t-h(t)}^{t}e(s)ds-\frac{12}{h^2(t)}\int_{t-h(t)}^{t}\int_{u}^{t}e(s)dsdu \end{bmatrix}^T$$
$$\times\begin{bmatrix} U_3 & 0 & 0 \\ 0 & 3U_3 & 0 \\ 0 & 0 & 5U_3 \end{bmatrix}$$
$$\times\begin{bmatrix} e(t)-e(t-h(t)) \\ e(t)+e(t-h(t))-\frac{2}{h(t)}\int_{t-h(t)}^{t}e(s)ds \\ e(t)-e(t-h(t))+\frac{6}{h(t)}\int_{t-h(t)}^{t}e(s)ds-\frac{12}{h^2(t)}\int_{t-h(t)}^{t}\int_{u}^{t}e(s)dsdu \end{bmatrix}$$
$$-\frac{\lambda_2}{\lambda_2-h(t)}\times$$

$$\begin{bmatrix} e(t-h(t))-e(t-\lambda_2) \\ e(t-h(t))+e(t-\lambda_2)-\frac{2}{\lambda_2-h(t)}\int_{t-\lambda_2}^{t-h(t)}e(s)ds \\ e(t-h(t))-e(t-\lambda_2)+\frac{6}{\lambda_2-h(t)}\int_{t-\lambda_2}^{t-h(t)}e(s)ds-\frac{12}{(\lambda_2-h(t))^2}\int_{t-\lambda_2}^{t-h(t)}\int_{u}^{t-h(t)}e(s)dsdu \end{bmatrix}^T$$
$$\times \begin{bmatrix} U_3 & 0 & 0 \\ 0 & 3U_3 & 0 \\ 0 & 0 & 5U_3 \end{bmatrix} \times$$
$$\begin{bmatrix} e(t-h(t))-e(t-\lambda_2) \\ e(t-h(t))+e(t-\lambda_2)-\frac{2}{\lambda_2-h(t)}\int_{t-\lambda_2}^{t-h(t)}e(s)ds \\ e(t-h(t))-e(t-\lambda_2)+\frac{6}{\lambda_2-h(t)}\int_{t-\lambda_2}^{t-h(t)}e(s)ds-\frac{12}{(\lambda_2-h(t))^2}\int_{t-\lambda_2}^{t-h(t)}\int_{u}^{t-h(t)}e(s)dsdu \end{bmatrix}$$
$$\leq -\chi^T(t)\Pi_3^T\Phi_2\Pi_3\chi(t). \tag{3.21}$$

Moreover, we have the following relations from Lemma 2.6

$$\begin{aligned}
&-(1-\gamma)\int_{t-h(t)}^{t}\int_{s}^{t}\dot{e}^T(s)G_1\dot{e}(s)dsd\theta \\
\leq\ & -2(1-\gamma)\left[e(t)-\frac{1}{h(t)}\int_{t-h(t)}^{t}e(s)ds\right]^T G_1\left[e(t)-\frac{1}{h(t)}\int_{t-h(t)}^{t}e(s)ds\right] \\
&-4(1-\gamma)\left[e(t)+\frac{2}{h(t)}\int_{t-h(t)}^{t}e(s)ds-\frac{6}{h^2(t)}\int_{t-h(t)}^{t}\int_{u}^{t}e(s)dsdu\right]^T G_1 \\
&\times\left[e(t)+\frac{2}{h(t)}\int_{t-h(t)}^{t}e(s)ds-\frac{6}{h^2(t)}\int_{t-h(t)}^{t}\int_{u}^{t}e(s)dsdu\right] \\
&-6(1-\gamma)\left[e(t)-\frac{3}{h(t)}\int_{t-h(t)}^{t}e(s)ds+\frac{24}{h^2(t)}\int_{t-h(t)}^{t}\int_{u}^{t}e(s)dsdu\right. \\
&\left.-\frac{60}{h^3(t)}\int_{t-h(t)}^{t}\int_{u}^{t}\int_{s}^{t}e(r)drdsdu\right]^T G_1 \\
&\times\left[e(t)-\frac{3}{h(t)}\int_{t-h(t)}^{t}e(s)ds+\frac{24}{h^2(t)}\int_{t-h(t)}^{t}\int_{u}^{t}e(s)dsdu\right. \\
&\left.-\frac{60}{h^3(t)}\int_{t-h(t)}^{t}\int_{u}^{t}\int_{s}^{t}e(r)drdsdu\right] \\
=\ & -(1-\gamma)\chi^T(t)\Pi_5^T\Theta_1\Pi_5\chi(t).
\end{aligned} \tag{3.22}$$

Similarly, we have

$$\begin{aligned}
-\ & (1-\beta)\int_{t-d(t)}^{t}\int_{s}^{t}\dot{e}^T(s)G_2\dot{e}(s)dsd\theta \\
\leq\ & -2(1-\beta)\left[e(t)-\frac{1}{d(t)}\int_{t-d(t)}^{t}e(s)ds\right]^T G_2\left[e(t)-\frac{1}{d(t)}\int_{t-d(t)}^{t}e(s)ds\right]
\end{aligned}$$

$$
\begin{aligned}
& -4(1-\beta)\left[e(t)+\frac{2}{d(t)}\int_{t-d(t)}^{t} e(s)ds-\frac{6}{d^2(t)}\int_{t-d(t)}^{t}\int_{u}^{t} e(s)dsdu\right]^T G_2 \\
& \times\left[e(t)+\frac{2}{d(t)}\int_{t-d(t)}^{t} e(s)ds-\frac{6}{d^2(t)}\int_{t-d(t)}^{t}\int_{u}^{t} e(s)dsdu\right] \\
& -6(1-\beta)\left[e(t)-\frac{3}{d(t)}\int_{t-d(t)}^{t} e(s)ds+\frac{24}{d^2(t)}\int_{t-d(t)}^{t}\int_{u}^{t} e(s)dsdu\right. \\
& \left.-\frac{60}{d^3(t)}\int_{t-d(t)}^{t}\int_{u}^{t}\int_{s}^{t} e(r)drdsdu\right]^T G_2 \\
& \times\left[e(t)-\frac{3}{d(t)}\int_{t-d(t)}^{t} e(s)ds+\frac{24}{d^2(t)}\int_{t-d(t)}^{t}\int_{u}^{t} e(s)dsdu\right. \\
& \left.-\frac{60}{d^3(t)}\int_{t-d(t)}^{t}\int_{u}^{t}\int_{s}^{t} e(r)drdsdu\right] \\
= & -(1-\beta)\chi^T(t)\Pi_6^T\Theta_2\Pi_6\chi(t). \qquad (3.23)
\end{aligned}
$$

From (2.11), it is not difficult to verify that for any ε_1, $\varepsilon_2>0$ the nonlinear functions $\tilde{\xi}(e(t))$ and $\tilde{\zeta}(e(t-h(t)))$ satisfy

$$
-\varepsilon_1\begin{bmatrix} e(t) \\ \tilde{\xi}(e(t))\end{bmatrix}^T\begin{bmatrix}\overline{R}_{\Lambda_1} & \overline{S}_{\Lambda_1} \\ * & I\end{bmatrix}\begin{bmatrix} e(t) \\ \tilde{\xi}(e(t))\end{bmatrix}\geq 0, \qquad (3.24)
$$

$$
-\varepsilon_2\begin{bmatrix} e(t-h(t)) \\ \tilde{\zeta}(e(t-h(t)))\end{bmatrix}^T\begin{bmatrix}\overline{R}_{\Lambda_2} & \overline{S}_{\Lambda_2} \\ * & I\end{bmatrix}\begin{bmatrix} e(t-h(t)) \\ \tilde{\zeta}(e(t-h(t)))\end{bmatrix}\geq 0, \qquad (3.25)
$$

where $\overline{R}_{\Lambda_1}, \overline{R}_{\Lambda_2}, \overline{S}_{\Lambda_2}$ and $\overline{S}_{\Lambda_2}$ are also defined in (3.1).
Then, it is clearly that for any appropriately dimensioned matrices Υ_1^T and Υ_2^T the following equation holds:

$$
\begin{aligned}
0 \;=\; & 2\left[e^T(t)\Upsilon_1^T+\dot{e}(t)\Upsilon_2^T\right]\Big[-\dot{e}(t)+\tilde{\xi}(e(t))+\tilde{\zeta}(e(t-h(t)))+c_1\left(A\otimes D_1\right)e(t) \\
& +c_2\left(B\otimes D_2\right)e(t-h(t))+Ke(t)+\overline{K}e(t-d(t))+\omega(t)\Big]. \qquad (3.26)
\end{aligned}
$$

Adding the right-hand sides of (3.24)-(3.26) to $\dot{V}(t)$, we can get from (3.8)-(3.23) that

$$
\dot{V}(t)+\sigma y^T(t)y(t)-2(1-\sigma)\delta y^T(t)\omega(t)-\delta^2\omega^T(t)\omega(t)\leq\chi^T(t)\Psi\chi(t). \qquad (3.27)
$$

Thus, according to Eq. (3.4) we have

$$
\dot{V}(t)+\sigma y^T(t)y(t)-2(1-\sigma)\delta y^T(t)\omega(t)-\delta^2\omega^T(t)\omega(t)<0. \qquad (3.28)
$$

Then, under the zero original condition, it can be inferred that for any $\mathcal{T}_p$

$$
\begin{aligned}
& \int_0^{\mathcal{T}_p}\sigma y^T(t)y(t)-2(1-\sigma)\delta y^T(t)\omega(t)-\delta^2\omega^T(t)\omega(t)\,dt \\
& \quad\leq\int_0^{\mathcal{T}_p}\dot{V}(t)+\sigma y^T(t)y(t)-2(1-\sigma)\delta y^T(t)\omega(t)-\delta^2\omega^T(t)\omega(t)\,dt<0,
\end{aligned}
$$

which indicates that

$$\int_0^{\mathcal{T}_p} \sigma y^T(t)y(t) - 2(1-\sigma)\delta y^T(t)\omega(t)\,dt \le \delta^2 \int_0^{\mathcal{T}_p} \omega^T(t)\omega(t)\,dt.$$

In this case, the condition (2.18) is assured for any non-zero $\omega(t) \in \mathcal{L}_2[0,\infty)$. If $\omega(t) = 0$, in view of equation (3.27), we have

$$\dot{V}(t) < -\sigma y^T(t)y(t) + \chi^T(t)\Psi\chi(t) \le -\sigma\lambda_{\min}\left(Z^T Z\right)\|e(t)\|^2 + \lambda_{\max}(\Psi)\|\dot{e}(t)\|^2. \tag{3.29}$$

Also, from the definitions of $V_i(t)$, it is not difficult to obtain the following inequalities:

$$\begin{aligned}
V_1(t) &\le \lambda_{\max}(P)\|e(t)\|^2,\\
V_4(t) &\le \int_{t-\lambda_2}^{t} e^T(s)T_3 e(s)\,ds,\\
V_6(t) &\le \lambda_1^2 \int_{t-\lambda_1}^{t} \dot{e}^T(s)U_1\dot{e}(s)\,ds,\\
V_7(t) &\le d^2 \int_{t-d}^{t} \dot{e}^T(s)U_2\dot{e}(s)\,ds,\\
V_8(t) &\le \lambda_2^2 \int_{t-\lambda_2}^{t} \dot{e}^T(s)U_3\dot{e}(s)\,ds,\\
V_9(t) &\le \lambda_2^2 \int_{t-\lambda_2}^{t} \dot{e}^T(s)U_4\dot{e}(s)\,ds,\\
V_{10}(t) &\le \frac{\lambda_2^2}{2} \int_{t-\lambda_2}^{t} \dot{e}^T(s)G_1\dot{e}(s)\,ds,\\
V_{11}(t) &\le \frac{d^2}{2} \int_{t-d}^{t} \dot{e}^T(s)G_2\dot{e}(s)\,ds.
\end{aligned} \tag{3.30}$$

We are now ready to deal with the exponential stability of (2.14) and (2.15). We consider the LyapunovKrasovskii functional $e^{2kt}V(t)$, where k is a constant to be determined. Using (3.29), (3.30), we have

$$\begin{aligned}
&\frac{d}{dt}e^{2kt}V(t)\\
&= e^{2kt}\dot{V}(t) + 2ke^{2kt}V(t)\\
&< e^{2kt}\Bigg[\Bigg(-\sigma\lambda_{\min}\left(Z^T Z\right) + 2k\Big(\lambda_{\max}(P) + \lambda_1\lambda_{\max}(T_1) + \lambda_2\lambda_{\max}(T_2) + \lambda_2\lambda_{\max}(T_3)\\
&\quad + d\lambda_{\max}(T_4)\Big)\Bigg) \sup_{-\max\{\lambda_2,d\}\le\theta\le 0} \|e(t+\theta)\|^2\\
&\quad + \Bigg(\lambda_{\max}(\Psi) + 2k\Big(\lambda_1^2\lambda_{\max}(U_1) + d^2\lambda_{\max}(U_2) + \lambda_2^2\lambda_{\max}(U_3) + \lambda_2^2\lambda_{\max}(U_4)\\
&\quad + \frac{\lambda_2^3}{2}\lambda_{\max}(G_1) + \frac{d^3}{2}\lambda_{\max}(G_2)\Big)\Bigg) \sup_{-\max\{\lambda_2,d\}\le\theta\le 0} \|\dot{e}(t+\theta)\|^2\Bigg].
\end{aligned} \tag{3.31}$$

Given

$$\mu_1 = \lambda_{\max}(P) + \lambda_1\lambda_{\max}(T_1) + \lambda_2\lambda_{\max}(T_2) + \lambda_2\lambda_{\max}(T_3) + d\lambda_{\max}(T_4)$$
$$\mu_2 = \lambda_1^2\lambda_{\max}(U_1) + d^2\lambda_{\max}(U_2) + \lambda_2^2\lambda_{\max}(U_3) + \lambda_2^2\lambda_{\max}(U_4) + \frac{\lambda_2^3}{2}\lambda_{\max}(G_1) + \frac{d^3}{2}\lambda_{\max}(G_2).$$

Set $k_0 = \min\left\{\frac{\sigma\lambda_{\min}\left(Z^TZ\right)}{2\mu_1}, \frac{-\lambda_{\max}\left(\Psi\right)}{2\mu_2}\right\}$.

From now on, we take k to be a constant satisfying $k \le k_0$ and then obtain from (3.31) that

$$\frac{d}{dt}e^{2kt}V(t) \le 0, \tag{3.32}$$

which, together with (3.8) and (3.30), imply that

$$\begin{aligned}
e^{2kt}V(t) \le V(0) &= \sum_{i=1}^{11} V_i(0)\\
&\le \Bigg[\lambda_{\max}(P)\|e(0)\|^2 + \int_{-\lambda_1}^{0} e^T(s)T_1e(s)\,ds + \int_{-\lambda_2}^{0} e^T(s)T_2e(s)\,ds\\
&+ \int_{-\lambda_2}^{0} e^T(s)T_3e(s)\,ds + \int_{-d}^{0} e^T(s)T_4e(s)\,ds + \lambda_1^2\int_{-\lambda_1}^{0} \dot{e}^T(s)U_1\dot{e}(s)\,ds\\
&+ d^2\int_{-d}^{0} \dot{e}^T(s)U_2\dot{e}(s)\,ds + \lambda_2^2\int_{-\lambda_2}^{0} \dot{e}^T(s)U_3\dot{e}(s)\,ds + \lambda_2^2\int_{-\lambda_2}^{0} \dot{e}^T(s)U_4\dot{e}(s)\,ds\\
&+ \frac{\lambda_2^2}{2}\int_{-\lambda_2}^{0} \dot{e}^T(s)G_1\dot{e}(s)\,ds + \frac{d^2}{2}\int_{-d}^{0} \dot{e}^T(s)G_2\dot{e}(s)\,ds\Bigg]\\
&\le \mu_0 \max\{\sup_{-\max\{\lambda_2,d\}\le\theta\le 0}\|e(\theta)\|^2, \sup_{-\max\{\lambda_2,d\}\le\theta\le 0}\|\dot{e}(\theta)\|^2\},
\end{aligned}$$

where

$$\begin{aligned}
\mu_0 = {} & \lambda_{\max}(P) + \lambda_1\lambda_{\max}(T_1) + \lambda_2\lambda_{\max}(T_2) + \lambda_2\lambda_{\max}(T_3) + d\lambda_{\max}(T_4)\\
& + \lambda_1^3\lambda_{\max}(U_1) + d^3\lambda_{\max}(U_2) + \lambda_2^3\lambda_{\max}(U_3) + \lambda_2^3\lambda_{\max}(U_4)\\
& + \frac{\lambda_2^3}{2}\lambda_{\max}(G_1) + \frac{d^3}{2}\lambda_{\max}(G_2),
\end{aligned}$$

therefore

$$V(t) \le \mu_0 e^{-2kt}\max\{\sup_{-\max\{\lambda_2,d\}\le\theta\le 0}\|e(\theta)\|^2, \sup_{-\max\{\lambda_2,d\}\le\theta\le 0}\|\dot{e}(\theta)\|^2\}.$$

By noticing $\lambda_{\min}(P)\|e(t)\|^2 \le V(t)$, we obtain

$$\|e(t)\|^2 \le \frac{\mu_0}{\lambda_{\min}(P)}e^{-2kt}\max\{\sup_{-\max\{\lambda_2,d\}\le\theta\le 0}\|e(\theta)\|^2, \sup_{-\max\{\lambda_2,d\}\le\theta\le 0}\|\dot{e}(\theta)\|^2\}. \tag{3.33}$$

By letting $\eta = \dfrac{\mu_0}{\lambda_{\min}(P)}$ and $\varpi = 2k$, we can rewrite (3.33) as

$$\|e(t)\|^2 \leq \eta e^{-\varpi t} \max\{\sup_{-\max\{\lambda_2,d\}\leq\theta\leq 0} \|e(\theta)\|^2, \sup_{-\max\{\lambda_2,d\}\leq\theta\leq 0} \|\dot{e}(\theta)\|^2\}.$$

Then, the system (2.14) and (2.15) are exponentially stable. Therefore, according to Definition 2.3, the system (2.14) and (2.15) are exponentially stable with a mixed H_∞ and passivity performance index δ. This completes the proof. ■

Theorem 3.2. *Consider scalars $\lambda_1, \lambda_2, \gamma, \beta, d, \alpha_1, \alpha_2, \varepsilon_1$ and ε_2, matrices R_1, R_2 S_1, and S_2. The complex network (2.1) and (2.2) are exponentially synchronized by the hybrid feedback controllers (2.9) if there exist matrices $P = \mathrm{diag}\{P_1, P_2, \ldots, P_N\} > 0$, $M = \mathrm{diag}\{M_1, M_2, \ldots, M_N\}$, $T_i > 0$, $U_i > 0$ $(i = 1, 2, 3, 4), G_1 > 0, G_2 > 0, \Sigma = \mathrm{diag}\{\Sigma_1, \Sigma_2, \ldots, \Sigma_N\}$, $\Omega = \mathrm{diag}\{\Omega_1, \Omega_2, \ldots, \Omega_N\}$,* $X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{bmatrix} \in \mathbb{R}^{3nN\times 3nN}, Y = \begin{bmatrix} Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & Y_6 \\ Y_7 & Y_8 & Y_9 \end{bmatrix} \in \mathbb{R}^{3nN\times 3nN}$ *such that (3.5), (3.6) and the following conditions hold:*

$$\begin{aligned}
\Xi \;=\;& 2r_1^T P r_2 + r_1^T T_1 r_1 - r_6^T T_1 r_6 + r_1^T T_2 r_1 - r_7^T T_2 r_7 + r_1^T T_3 r_1 - (1-\gamma)\, r_3^T T_3 r_3 \\
& + r_1^T T_4 r_1 - r_5^T T_4 r_5 + \lambda_1^2 r_2^T U_1 r_2 - \Pi_1^T U_1 \Pi_1 + d^2 r_2^T U_2 r_2 - \Pi_2^T U_2 \Pi_1 \Pi_2 \\
& + \lambda_2^2 r_2^T U_3 r_2 - \Pi_3^T \Phi_2 \Pi_3 + \lambda_2^2 r_2^T U_4 r_2 - (1-\gamma)\, \Pi_4^T U_4 \Pi_4 + \frac{\lambda_2^2}{2} r_2^T G_1 r_2 \\
& -(1-\gamma)\Pi_5^T \Theta_1 \Pi_5 + \frac{d^2}{2} r_2^T G_2 r_2 - (1-\beta)\Pi_6^T \Theta_2 \Pi_6 - 2r_1^T \alpha_1 M^T r_2 \\
& + 2r_1^T \alpha_1 M^T r_{18} + 2r_1^T \alpha_1 M^T r_{19} + 2r_1^T \alpha_1 M^T c_1 (A \otimes D_1) r_1 \\
& + 2r_1^T \alpha_1 M^T c_2 (B \otimes D_2) r_3 + 2r_1^T \alpha_1 \Sigma r_1 + 2r_1^T \alpha_1 \Omega r_4 - 2r_2^T \alpha_2 M^T r_2 \\
& + 2r_2^T \alpha_2 M^T r_{18} + 2r_2^T \alpha_2 M^T r_{19} + 2r_2^T \alpha_2 M^T c_1 (A \otimes D_1) r_1 \\
& + 2r_2^T \alpha_2 M^T c_2 (B \otimes D_2) r_3 + 2r_2^T \alpha_2 \Sigma r_1 + 2r_2^T \alpha_2 \Omega r_4 + 2r_1^T \alpha_1 M^T r_{20} \\
& + 2r_2^T \alpha_2 M^T r_{20} - \varepsilon_1 \Pi_7^T \begin{bmatrix} \overline{R}_{\Lambda_1} & \overline{S}_{\Lambda_1} \\ * & I \end{bmatrix} \Pi_7 - \varepsilon_2 \Pi_8^T \begin{bmatrix} \overline{R}_{\Lambda_2} & \overline{S}_{\Lambda_2} \\ * & I \end{bmatrix} \Pi_8 \\
& + r_1^T \sigma Z^T Z r_1 - 2(1-\sigma)\delta r_1^T Z^T r_{20} - r_{20}^T \delta^2 I r_{20} < 0. \qquad (3.34)
\end{aligned}$$

Moreover, if the LMIs (3.34) is solvable, the desired controllers gain matrices are given as

$$K = M^{-1}\Sigma, \qquad \overline{K} = M^{-1}\Omega. \qquad (3.35)$$

Proof. Denote

$$\Upsilon_1 = \mathrm{diag}\{\alpha_1 M_1, \alpha_1 M_2, \ldots, \alpha_1 M_N\},\ \Upsilon_2 = \mathrm{diag}\{\alpha_2 M_1, \alpha_2 M_2, \ldots, \alpha_2 M_N\}.$$

Then we can obtain (3.34). This completes the proof. ■

4. Numerical Example

In this section, a numerical example is adopted to valid our synchronization controller of the complex network (2.1), (2.2).

Example 4.1. Consider a complex network (2.1) and (2.2) with three nodes. The outer-coupling matrix is assumed to be $A = (a_{ij})_{3\times 3}, B = (b_{ij})_{3\times 3}$ with

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

The inner-coupling matrix is given as $D_1 = \mathrm{diag}\{0.4, 0.4\}, D_2 = \mathrm{diag}\{0.5, 0.5\}$ and the time-varting delay is chosen as $h(t) = 1.95 + 0.05\sin(10t)$. Accordingly, we have $\lambda_1 = 0.1,\ \lambda_2 = 2,\ \gamma = 0.5$ and we choose $\omega(t) = \frac{0.2}{1+t^2}, Z = I_3 \otimes I_2,\ \sigma = 0.5,\ \delta = 0.2$ and the following parameters

$$c_1 = 1,\ c_2 = 1,\ d = 0.5,\ \beta = 0.5,\ \varepsilon_1 = 0.9,\ \varepsilon_2 = 0.7,\ \alpha_1 = 0.9,\ \alpha_2 = 0.7.$$

The nonlinear functions $f(\cdot),\ g(\cdot)$ are taken as

$$f(x_i(t)) = \begin{bmatrix} -0.5x_{i1}(t) + \tanh(0.2x_{i1}(t)) + 0.2x_{i2}(t) \\ 0.95x_{i2}(t) - \tanh(0.75x_{i2}(t)) \end{bmatrix}, \quad i = 1, 2, 3.$$

$$g(x_i(t-h(t))) = \begin{bmatrix} -0.5x_{i1}(t-h(t)) + \tanh(0.2x_{i1}(t-h(t))) + 0.2x_{i2}(t-h(t)) \\ 0.45x_{i2}(t-h(t)) - \tanh(0.25x_{i2}(t-h(t))) \end{bmatrix}.$$

It is easy to see that $f(\cdot)$ and $g(\cdot)$ satisfy (2.11) with

$$R_1 = \begin{bmatrix} -0.5 & 0.2 \\ 0 & 0.95 \end{bmatrix}, \quad S_1 = \begin{bmatrix} -0.3 & 0.2 \\ 0 & 0.2 \end{bmatrix},$$
$$R_2 = \begin{bmatrix} -0.5 & 0.2 \\ 0 & 0.45 \end{bmatrix}, \quad S_2 = \begin{bmatrix} -0.3 & 0.2 \\ 0 & 0.2 \end{bmatrix}.$$

By applying Theorem 3.2, and solving LMIs (3.34), the contoller gain matrices can be obtained as follows:

$$K_1 = \begin{bmatrix} -46.7963 & -0.1141 \\ -0.1276 & -38.5714 \end{bmatrix}, \quad \overline{K}_1 = \begin{bmatrix} -0.0132 & 0.0000 \\ 0.0000 & -0.0074 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -49.3467 & -0.1122 \\ -0.1194 & -42.3202 \end{bmatrix}, \quad \overline{K}_2 = \begin{bmatrix} 0.0024 & 0.0000 \\ 0.0001 & -0.0124 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} -41.6409 & 0.0571 \\ 0.0578 & -38.5915 \end{bmatrix}, \quad \overline{K}_3 = \begin{bmatrix} -0.0112 & 0.0000 \\ 0.0000 & -0.0090 \end{bmatrix}.$$

In the simulation, the parameters of initial values are settled as $x_1(0) = [3\ -2]^T, x_2(0) = [2\ 5]^T, x_3(0) = [-5\ 6]^T, s(0) = [-1\ 2]^T$. The state trajectories of the controlled complex network are shown in Figure 1. and 2. From our simulation results, it can be seen that the designed hybrid feedback controllers achieve the exponential synchronization of the complex network (2.1) and (2.2).

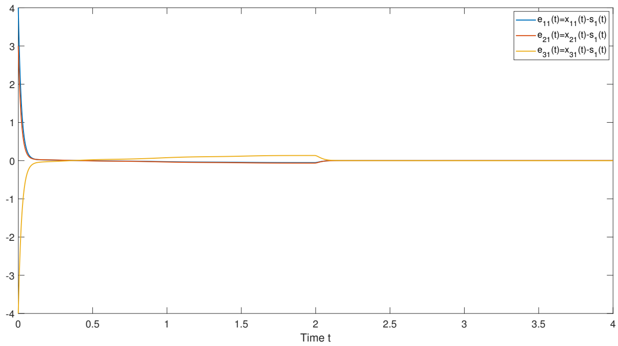

FIGURE 1. Error state $e_{i1}(t)$ of the controlled complex network $(i = 1, 2, 3)$.

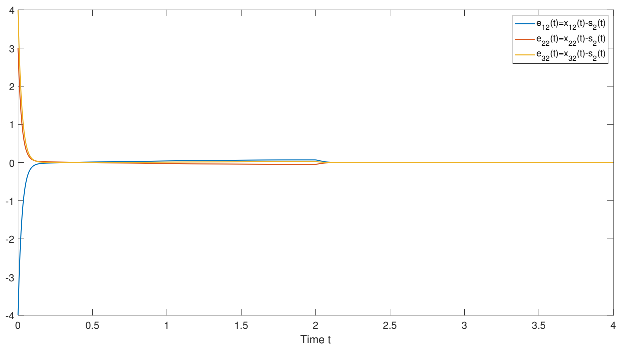

FIGURE 2. Error state $e_{i2}(t)$ of the controlled complex network $(i = 1, 2, 3)$.

5. Conclusions

In this paper, we focus on the problem of mixed H_∞ and passive synchronization for complex dynamical networks with time-varying delay and mixed coupling. By designing hybrid feedback controller and constructing a Lyapunov-Krasovskii functional is based on Jensens inequality, relaxed double integral inequality and improved reciprocally convex approach, the sufficient conditions guaranteed mixed H_∞ and passive performance level for CDNs with time-varying delay. These results have been expressed in terms of linear matrix inequalities (LMIs). Finally, a numerical example demonstrates the effectiveness of the proposed method.

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