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# GENERALIZED HADAMARD WELLPOSED FOR LEXICOGRAPHIC VECTOR EQUILIBRIUM PROBLEMS

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**Abstract** In this paper, we study Painlevé-Kuratowski convergence of the solution sets with a sequence of mappings converges continuously. By considering the solution sets of lexicographic vector equilibrium problems, we establish necessary and/or sufcient conditions to be Hadamard well-posed for the mentioned problems in the sense of Painlevé-Kuratowski. The results in this paper unified, generalized and extended some known results in the literature. By obtaining consequences of the results, we also discuss lexicographic variational inequalities as an application.

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**Keywords:** lexicographic vector equilibrium problems; Hadamard well-posedness; sequence of sets; sequence of mappings; Painlevé-Kuratowski convergence; lexicographic variational inequalities.

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## 1. Introduction

Equilibrium problems first considered by Blum and Oettli [12] have been playing an important role in optimization theory with many striking applications particularly in transportation, mechanics, economics, etc. Equilibrium models incorporate many other important problems such as: optimization problems, variational inequalities, complementarity problems, saddlepoint/minimax problems, and fixed points. Equilibrium problems with scalar and vector objective functions have been widely studied. The crucial issue of solvability (the existence of solutions) has attracted the most considerable attention of researchers, see, e.g., [9, 14, 18]. A relatively new but rapidly growing topic is the stability of solutions, including semicontinuity properties in the sense of Berge and Hausdorff, see, e.g., [4, 7] and the Hölder/Lipschitz continuity of solution mappings, see, e.g., [1, 5, 6, 8].

Most works on vector variational inequality and vector equilibrium problems are based on orders induced by convex closed cones, i.e., they used various extensions of the Pareto order. However, it is known from the theory of vector optimization that the set of Pareto-optimal points is usually too large, so that one needs certain additional rules to reduce it. One of the possible approaches is to utilize the lexicographic order, which was investigated in connection with its applications in optimization and decision making theory. Konnov

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[21] studied vector equilibrium problems using the lexicographic order and obtained that several classes of inverse lexicographic optimization problems can be reduced to lexicographic vector equilibrium problems. In [10] Bianchi et al. investigated equivalence properties between various kinds of lexicographic variational inequalities and obtained several existence results for lexicographic variational inequalities. In [11] Bianchi et al. analyzed lexicographic equilibrium problems on a topological Hausdorff vector space, their relationships with some other vector equilibrium problems, and existence results for the tangled lexicographic problem are proved via the study of a related sequential problem. Fang et al. [17] studied parametric vector equilibrium problem in a lexicographic order. They obtained the lower semicontinuity of the solution set map based on the density of the solution set mapping for a parametric lexicographic vector equilibrium problem by using an auxiliary problem. Anh et al. [2] investigated lexicographic vector equilibrium problems in metric spaces and established sufficient conditions for a family of such problems to be (uniquely) well-posed at the reference point. Recently, Anh et al. [3] established the sufficient conditions for the upper semicontinuity, closedness, and continuity of solution maps for a parametric lexicographic equilibrium problem. Very recently, Rabian et al. studied the well-posedness for lexicographic vector equilibrium problems and optimization problems with lexicographic equilibrium constraints in metric spaces and obtained sufficient conditions for a family of such problems to be (uniquely) well-posed at the reference point.

Well-posedness of optimization-related problems can be defined in two ways. The first and oldest is Hadamard well-posedness [19], which means existence, uniqueness and continuous dependence of the optimal solution and optimal value from perturbed data. The second is Tikhonov well-posedness [26], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Well-posedness properties have been intensively studied and the two classical well-posedness notions have been extended and blended. Recently, the Tikhonov notion has been more interested. The major reason is its vital role in numerical methods. Any algorithm can generate only an approximating sequence of solutions. Hence, this sequence is applicable only if the problem under consideration is well-posed. For parametric problems, well-posedness is closely related to stability.

As for the stabile results investigated on the convergence of the sequence of mappings, there are some results for the vector optimization, vector variational inequality problems and vector equilibrium problems with a sequence of sets converging in the sense of Painlevé-Kuratowski (see e.g., [15, 16, 20, 22, 24]). In [20], Huang discussed the convergence of the approximate efficient sets to the efficient sets of vector-valued and set-valued optimization problems in the sense of Painlevé-Kuratowski and Mosco. In [16], Fang et al. investigated the Painlevé-Kuratowski convergence of the solution sets of the perturbed set-valued weak vector variational inequality problems. In [22], Lalitha and Chatterjee investigated the Painlevé-Kuratowski set convergence of the solution sets of a nonconvex vector optimization problem. In [24], Peng and Yang investigated the Painlevé-Kuratowski set convergence of the solution sets of the perturbed vector equilibrium problems without monotonicity in real linear metric spaces. Very recently, Li et al. [23] concerned with the stability for a generalized Ky Fan inequality when it is perturbed by vector-valued bifunction sequence and set sequence. By continuous convergence of the bifunction sequence and Painlevé-Kuratowski convergence of the set sequence, they established the Painlevé-Kuratowski convergence of the approximate solution mappings of a family of perturbed problems to the corresponding solution mapping of the original problem.

Motivated and inspired by research work mentioned above, in this paper, we study the lexicographic vector equilibrium problems and their relationships. Namely, we study the sufficient conditions for a family of such problems to be generalized Hadamard well-posed in the sense of Painlevé-Kuratowski. By using some technique, the inner convergence of sequence solution sets for a lexicographic vector equilibrium problem are established and then we establish Painlevé-Kuratowski convergence of the solution sets with a sequence of mappings converging continuously and sequence of set converging in the sense of Painlevé-Kuratowski. In addition, we also consider particular cases lexicographic variational inequalities and lexicographic optimization problems.

This paper is organized as follows. Section 2 presents some necessary notations and definitions. In Section 3, the concepts of generalized Hadamard well-posed in the sense of Painlevé-Kuratowski for lexicographic vector equilibrium problem (LEP) are introduced. Section 4 contains some important particular cases as examples of applications of our results.

## 2. Preliminaries

**Definition 2.1.** Let $g$ be an extended real-valued function on a metric space $X$ and $\varepsilon$ be a real number. $g$ is upper $\varepsilon$-level closed at $x_0$ if for any sequence $\{x_n\} \subseteq X$, $x_n \to x_0$,

$$[g(x_n) \geq \varepsilon, \forall n] \Rightarrow [g(x_0) \geq \varepsilon].$$

**Definition 2.2.** Let $X$ and $Y$ be two metric spaces and $G : X \to 2^Y$ be a set-valued mapping.

(i) $G$ is said to be *lower semicontinuous* at $x_0 \in X$, if $G(x_0) \cap U \neq \emptyset$ for some open set $U \subset Y$ implies the existence of a neighborhood $N$ of $x_0$ such that $G(x) \cap U \neq \emptyset, \forall x \in N$. $G$ is said to be lower semicontinuous in $X$ if it is lower semicontinuous at each $x_0 \in X$.

(ii) $G$ is said to be *upper semicontinuous* at $x_0 \in X$, if for each open set $U \supseteq G(x_0)$, there is a neighborhood $N$ of $x_0$ such that $U \supseteq G(x), \forall x \in N$. $G$ is said to be upper semicontinuous in $X$ if it is upper semicontinuous at each $x_0 \in X$.

(iii) $G$ is said to be *continuous* at $x_0 \in X$, if it is both lower semicontinuous and upper semicontinuous at $x_0$. $G$ is said to be continuous in $X$ if it is both lower semicontinuous and upper semicontinuous at each $x_0 \in X$.

(vi) $G$ is said to be *closed* at $x_0$, if for each sequence $\{(x_n, y_n)\} \subset \text{graph}G := \{(x,y)|y \in G(x)\}, (x_n, y_n) \to (x_0, y_0)$, it follows that $(x_0, y_0) \in \text{graph}G$. $G$ is said to be closed in $X$ if it is closed at each $x_0 \in X$.

**Lemma 2.3.** *Let $X$ and $Y$ be two metric spaces and $G : X \to 2^Y$ be a set-valued mapping. If $G$ has compact values, then $G$ is upper semicontinuous at $x_0$ if and only if, for each sequence $\{x_n\} \subset X$ which converges to $x_0$ and for each sequence $\{y_n\} \subset G(x_n)$, there are $y \in G(x)$ and a subsequence $\{y_m\}$ of $\{y_n\}$ such that $y_m \to y$.*

**Lemma 2.4.** *Let $X$ and $Y$ be topological spaces. If a set-valued mapping $T : X \to 2^Y$ is upper semicontinuous with compact values, then for every compact set $K \subset X$, the set $T(K) = \cup_{x \in K} T(x)$ is compact.*

**Definition 2.5.** [25] Let $\{C_n\}$ be a sequence of sets of $\mathbb{R}^m$ and $C$ be a subset of $\mathbb{R}^m$.

(i) $\limsup_n C_n := \{x \in \mathbb{R}^m \mid \exists x_{n_k} \in C_{n_k}, x_{n_k} \to x\}$ is its outer limit;

(ii) $\liminf_n C_n := \{x \in \mathbb{R}^m \mid \exists x_n \in C_n, x_n \to x\}$ is its inner limit;

(iii) $\{C_n\}$ is said to be Painlevé-Kuratowski convergent to $C$, denoted by $C_n \xrightarrow{P.K.} C$, if and only if $\limsup_n C_n \subseteq C \subseteq \liminf_n C_n$.

The relations $\limsup_n C_n \subset C$ and $C \subset \liminf_n C_n$ are, respectively, referred as the upper part and the lower part of the convergence. Clearly, $\liminf_n C_n \subset \limsup_n C_n$.

**Definition 2.6.** [25] Let $S : X \to 2^Y$ be a set-valued mapping.

(i) $S$ is *outer semicontinuous (osc)* at $\bar{x}$ if $\limsup_{x\to\bar{x}} S(x) \subset S(\bar{x})$ with $\limsup_{x\to\bar{x}} S(x) := \cup_{x_n\to\bar{x}} \limsup_{n\to\infty} S(x_n)$.

(ii) $S$ is *inner semicontinuous (isc)* at $\bar{x}$ if $S(\bar{x}) \subset \liminf_{x\to\bar{x}} S(x)$ with $\liminf_{x\to\bar{x}} S(x) := \cap_{x_n\to\bar{x}} \liminf_{n\to\infty} S(x_n)$.

(iii) $S$ is said to be *continuous* at $\bar{x}$, written as $S(x) \to S(\bar{x})$ as $x \to \bar{x}$ if it is both outer semicontinuous and inner semicontinuous.

**Definition 2.7.** [25]A sequence of vector-valued bifunctions $\{f_n : K_n \times K_n \to \mathbb{R}^l\}$ converges continuously to vector-valued bifunction $f : K \times K \to \mathbb{R}^l$ and is denoted by $(K_n, f_n) \xrightarrow{c} (K, f)$ if and only if $K_n \xrightarrow{P.K.} K$ and for any sequence $\{(x_n, y_n)\}$ in $K_n \times K_n$ converging to $(x, y)$,

$$f_n(x_n, y_n) \to f(x, y).$$

**Definition 2.8.** Let $A$ be a convex subset of $X$. A function $f : A \to \mathbb{R}$ is said to be *strictly concave* if for all $x, y \in A, x \neq y$ and $t \in (0, 1)$, one has

$$f(tx + (1-t)y) > tf(x) + (1-t)f(y).$$

Let $\{K_n\}$ be a sequence of sets of $\mathbb{R}^m$ and $K$ be a subset of $\mathbb{R}^m$. When $\lim_{n\to\infty} K_n$ exists in the sense of Painleavé-Kuratowski and equals $K$, the sequence $\{K_n\}_{n\in\mathbb{N}}$ is said to *converge* to $K$, written

$$K_n \xrightarrow{P.K.} K.$$

Set convergence in this sense is known more specifically as Painlevé-Kuratowski convergence. The following results investigate some properties of the sequence of sets when it converge in the sense of Painlevé-Kuratowski.

**Lemma 2.9.** *Assume that $K_n \xrightarrow{P.K.} K$.*
(i) *If $K_n$ is convex then $K$ is convex;*
(ii) *If $K_n$ is bounded then $K$ is bounded;*
(iii) *If $K_n$ is compact then $K$ is compact.*

*Proof.* (i) Let $x, y \in K$, $\lambda \in [0, 1]$. We claimed that $\lambda x + (1-\lambda)y \in K$, $\forall \lambda \in [0, 1]$. Since $K_n \xrightarrow{P.K.} K$ and $x, y \in K$, there exist sequences $\{x_n\}$, $\{y_n\}$ in $K_n$ converging to $x, y$, respectively. Thanks to the convexity of $K_n$, one has $\lambda x_n + (1-\lambda)y_n \in K_n$. So,

$$\inf d(\lambda x_n + (1-\lambda)y_n, K_n) = 0.$$

Then,

$$\lambda x_n + (1-\lambda)y_n \in \limsup_{n\to\infty} K_n.$$

One obtains that

$$\lambda x + (1-\lambda)y \in \limsup_{n\to\infty} K_n,$$

as $\limsup_{n\to\infty} K_n$ is closed. By $\limsup_{n\to\infty} K_n \subset K$, one has $\lambda x + (1-\lambda)y \in K$. So $K$ is convex.

(ii) Suppose to the contrary that $K$ is not bounded. Then for any positive real $r$, there exist $x, y \in K$ satisfying $d(x, y) > r$. Due to $K_n \xrightarrow{P.K.} K$ and $x, y \in K$, there exist sequences $\{x_n\}$, $\{y_n\}$ in $K_n$ converge to $x, y$, respectively. For each $i = \overline{1, n}$, one has

$$d(x, y) \leq d(x, x_i) + d(x_i, y_i) + d(y_i, y).$$

Taking limit two sides, one obtains that $d(x_i, y_i) > r$. So, $K_i$ is not bounded, which contradicts with the above assumption.

(iii) By (ii) and the closedness of $K$, we obtain (iii). ■

We next recall the concept of lexicographic cone in finite dimensional spaces and models of equilibrium problems with the order induced by such a cone. The lexicographic cone of $\mathbb{R}^n$, denoted $C_l$, is the collection of zero and all vectors in $\mathbb{R}^n$ with the first nonzero coordinate being positive, i.e.,

$$C_l := \{0\} \cup \{x \in \mathbb{R}^n | \exists i \in \{1, \ldots, n\} \text{ s.t. } x_i > 0 \text{ and } x_j = 0, \forall j < i\}.$$

This cone is convex and pointed, and induces the total order as follow:

$$x \geq_L y \Leftrightarrow x - y \in C_l.$$

We also observe that it is neither closed nor open. Indeed, when comparing with the cone $C_1 := \{x \in \mathbb{R}^n | x_1 \geq 0\}$, we see that $intC_1 \subsetneq C_l \subsetneq C_1$, while

$$intC_l = intC_1 \text{ and } clC_l = C_1.$$

Let $f = (f_1, f_2) : K \times K \to \mathbb{R}^2$ is a vector-valued function. Let us now consider the following lexicographic vector equilibrium problem in the space $\mathbb{R}^2$:

(LEP) find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq_L 0, \forall y \in K. \tag{2.1}$$

The set of solutions of the problems will be denoted by $S_{LEP}$.

For each $n \in \mathbb{N}$, let $f_n : K \times K \to \mathbb{R}$. We consider the following sequence of the lexicographic vector equilibrium problem:

$(\text{LEP})_n$ find $x_n \in K_n$ such that

$$f_n(x_n, y_n) \geq_L 0, \forall y_n \in K_n. \tag{2.2}$$

The set of solutions of the problems will be denoted by $S^n_{LEP}$. In this work, we always assume that $S_{LEP}$ and $S^n_{LEP}$ are nonempty sets.

It is worth noticing that (LEP) can be written in the following equivalent way:

(LEP) find $\bar{x} \in K$ such that

$$\begin{cases} f_1(\bar{x}, y) \geq 0, \forall y \in K; \\ f_2(\bar{x}, z) \geq 0, \forall z \in Z(\bar{x}). \end{cases} \tag{2.3}$$

where $Z : S_{EP_1(K)} \to 2^K$, $Z(x) = \{y \in K : f(x,y) = 0\}$ and $EP_1(K)$: find $\bar{x} \in K$ such that

$$f_1(\bar{x}, y) \geq 0, \forall y \in K. \tag{2.4}$$

Similarly, $(\text{LEP})_n$ can be presented in the following equivalent way:

$(\text{LEP})_n$ find $x_n \in K_n$ such that

$$\begin{cases} f_n^1(x_n, y_n) \geq 0, \forall y_n \in K_n; \\ f_n^2(x_n, z_n) \geq 0, \forall z_n \in Z_n(x_n). \end{cases} \tag{2.5}$$

where $Z_n : S_{EP_1(K)} \to 2^K$ is defined by

$$Z_n(x_n) = \{y_n \in K_n : f_n^1(x_n, y_n) = 0\}. \tag{2.6}$$

## 3. Main results

In this section, the concepts of generalized Hadamard well-posed in the sense of Painlevé-Kuratowski for (LEP) are introduced and their sufficient criteria are proposed.

Picking up the ideas in [3] , we first introduce the following lemma to ensure the lower semicontinuity of mapping $Z$.

**Lemma 3.1.** *Suppose that $f^1$ is continuous; the Fréchet derivative of $f^1$ with respect to the second argument exists and $D_2 f^1(x,y)$ is surjective for all $x, y \in X, x \neq y$. Then $Z$ is lsc.*

*Proof.* Suppose that there is $\bar{z} \in Z(\bar{x}) \setminus \{\bar{x}\}$. We show that there exists a sequence of points $z_n \in Z_n(x_n)$ with $z_n \to \bar{z}$. To see this, we claim that for each neighborhood $V$ of $\bar{z}$, $V \subseteq X$, there exist a neighborhood $U$ of $\bar{x}$ and a function $s_n : U \to V$ such that $s_n(x) \in Z_n(x)$ for all $x \in U$. Let $m = \left\| D_2 f^1(\bar{x}, \bar{z})^{-1} \right\|$ and $\alpha$ be a positive real number such that $B_\alpha(\bar{z}) \subseteq V$. From assumptions on $f^1$, one can choose a real number $\beta \in (0, \alpha]$ such that

$$\left| f^1(\bar{x}, z) - f^1(\bar{x}, \bar{z}) - \left\langle D_2 f^1(\bar{x}, \bar{z}), z - \bar{z} \right\rangle \right| \leq \frac{1}{2m} \left\| z - \bar{z} \right\|, \text{ for all } z \in B_\beta(\bar{z}),$$

and consequently,

$$\left| f_1(\bar{x}, z) - \langle D_2 f_1(\bar{x}, \bar{z}), z - \bar{z} \rangle \right| \leq \frac{\beta}{2m}, \text{ for all } z \in B_\beta(\bar{z}).$$

Since $f^1$ is continuous in $B_\alpha(\bar{x}) \times B_\alpha(\bar{z})$ and $f^1_{(n)} \to f^1$, there exists a positive real number $\gamma \leq \beta$ such that for all $x \in B_\gamma(\bar{x}), z \in B_\beta(\bar{z})$, one has

$$\left| f^1(x, z) - f^1(\bar{x}, z) \right| \leq \frac{\beta}{4m},$$

and

$$\left| f^1_{(n)}(x, z) - f^1(x, z) \right| \leq \frac{\beta}{4m}, \text{ for } n \text{ sufficiently large.}$$

These two inequalities give us

$$\left| f^1_{(n)}(x, z) - f^1(\bar{x}, z) \right| \leq \left| f^1_{(n)}(x, z) - f^1(x, z) \right| + \left| f^1(x, z) - f^1(\bar{x}, z) \right| \leq \frac{\beta}{2m}.$$

For each $x \in B_\gamma(\bar{x})$, we consider the function $\xi_x^{(n)} : B_\beta(\bar{z}) \to X$ defined by

$$\xi_x^{(n)}(z) = D_2 f^1(\bar{x}, \bar{z})^{-1} \left( \langle D_2 f^1(\bar{x}, \bar{z}), z \rangle - f^1_{(n)}(x, z) \right).$$

It is clear that $\xi_x^{(n)}$ is continuous in $B_\beta(\bar{z})$. Further, for any $z \in B_\beta(\bar{z})$, one has

$$\begin{aligned}
\left\| \xi_x^{(n)}(z) - \bar{z} \right\| &= \left\| D_2 f^1(\bar{x}, \bar{z})^{-1} \left( \langle D_2 f^1(\bar{x}, \bar{z}), z \rangle - f^1_{(n)}(x, z) \right) - \bar{z} \right\| \\
&= \left\| D_2 f^1(\bar{x}, \bar{z})^{-1} \right\| \left| \langle D_2 f^1(\bar{x}, \bar{z}), z - \bar{z} \rangle - f^1_{(n)}(x, z) \right| \\
&\leq m \left( \left| \langle D_2 f^1(\bar{x}, \bar{z}), z - \bar{z} \rangle - f^1(\bar{x}, z) \right| + \left| f^1(\bar{x}, z) - f^1_{(n)}(x, z) \right| \right) \\
&\leq m \left( \frac{\beta}{2m} + \frac{\beta}{2m} \right) = \beta.
\end{aligned}$$

Hence, thanks to the Brouwer's fixed-point theorem, for all $x \in B_\gamma(\bar{x})$, there exists a point $s_n(x) \in B_\beta(\bar{z}) \subseteq V$ such that $\xi_{(x)}^{(n)}(s_n(x)) = s_n(x)$. Thus,

$$s_n(x) = D_2 f^1(\bar{x}, \bar{z})^{-1} \left( \langle D_2 f^1(\bar{x}, \bar{z}), s_n(x) \rangle - f^1_{(n)}(x, s_n(x)) \right),$$

which is equivalent to $f^1_{(n)}(x, s_n(x)) = 0$, i.e. $s_n(x) \in Z_n(x)$.

∎

We now focus on the Painlevé-Kuratowski upper convergence and the closedness of the solution sets.

**Theorem 3.2.** *Suppose that*

(i) *$K_n$ converges to $K$;*
(ii) *$f_n^1(\cdot, \cdot)$ converges continuously to $f^1$ and upper 0-level closed at $(x_0, y_0)$, $f_n^1(x, \cdot)$ Fréchet differentiable with respect to $y$, $D_y f_n^1(x, y)$ is surjective for all $x \neq y$;*
(iii) *$f_n^2(\cdot, \cdot)$ is converges continuously to $f^2(\cdot, \cdot)$ and upper 0-level closed at $(x_0, y_0)$.*

*Then, $S^n_{LEP}$ is closed and*

$$\limsup_{n \to \infty} S^n_{LEP} \subseteq S_{LEP}.$$

*Proof.* Without loss of generality, we suppose that $n = 1$. We prove that $S^1_{LEP}$ is closed. Suppose to the contrary that there are sequences $\{x_m\} \subseteq S^1_{LEP}$ satisfying $x_m \to x_0$ but $x_0 \notin S^1_{LEP}$. Note that $x_0 \in K_1$ because $K_1$ is closed. Then, there exists $y_0 \in K_1$ satisfying

$$f_1^1(x_0, y_0) < 0, \tag{3.1}$$

and there exists $z_0 \in Z_1(x_0)$ such that

$$f_1^2(x_0, z_0) < 0. \tag{3.2}$$

As $y_0 \in K_1 \subseteq \overline{K_1}$, there exists a sequence $\{y_m\} \subseteq K_1$ such that $y_m \to y_0$. Since $z_0 \in Z_1(x_0)$ and $Z_1(\cdot)$ is inner semicontinuous at $x_0$, one has $z_0 \in \liminf_{n \to \infty} Z_1(x_m)$. Then, we can find a sequence $\{z_m\}$ in $Z_1(x_m)$ such that $z_m \to z_0$. Since $x_m \in S^1_{LEP}$, we have $f_1^1(x_m, y_m) \geq 0$ and $f_1^2(x_m, z_m) \geq 0$, respectively. By using (ii) and (iii), we obtain that $f_1^1(x_0, y_0) \geq 0$ and $f_1^2(x_0, z_0) \geq 0$, respectively. This is a contradiction to (3.1) and (3.2). Thus, we can conclude that $x_0 \in S^1_{LEP}$ and $S^1_{LEP}$ is closed.

Next, we prove that $\limsup_{n\to\infty} S^n_{LEP} \subseteq S_{LEP}$. Let $x_0 \in \limsup_{n\to\infty} S^n_{LEP}$. Then, there exists a subsequence $\{x_{n_k}\}$ in $S^{n_k}_{LEP}$ such that

$$x_{n_k} \to x_0 \text{ as } k \to \infty, \tag{3.3}$$

one has $x_0 \in \limsup_{n\to\infty} K_n$. As $K_n$ outer converges continuously to $K$, we have $x_0 \in K$. Let $y \in K$. We prove that

$$f(x_0, y) \geq_L 0.$$

As $\{x_{n_k}\} \subseteq S^{n_k}_{LEP}$, we have

$$f^1_{n_k}(x_{n_k}, y_{n_k}) \geq 0, \ \forall y_{n_k} \in K_{n_k} \tag{3.4}$$

and

$$f^2_{n_k}(x_{n_k}, z_{n_k}) \geq 0, \ \forall z_{n_k} \in Z_{n_k}(x_{n_k}). \tag{3.5}$$

Because $K_n$ inner converges continuously to $K$, it implies $y \in \liminf_{n\to\infty} K_n$. Then, we can find a sequence $y_n$ in $K_n$ such that $y_n \to y$. Clearly, $y_{n_k} \in K_{n_k}$ and

$$y_{n_k} \to y \text{ as } k \to \infty. \tag{3.6}$$

By (3.3), (3.6), and $f^1_{n_k}$ converges continuously to $f^1$, we get that

$$\limsup_{k\to\infty} f^1_{n_k}(x_{n_k}, y_{n_k}) \subseteq f^1(x_0, y) \subseteq \liminf_{k\to\infty} f^1_{n_k}(x_{n_k}, y_{n_k}).$$

So, the limit of $f^1_{n_k}(x_{n_k}, y_{n_k})$ exists and $\lim_{k\to\infty} f^1_{n_k}(x_{n_k}, y_{n_k}) = f^1(x_0, y)$. From (3.4), we have $f^1(x_0, y) \geq 0$. Next, we prove that $f^2(x_0, z) \geq 0$, $\forall z \in Z(x_0)$. Let $z \in Z(x_0)$. As $Z_n(\cdot)$ is inner converges continuously to $Z(\cdot)$ , one has $z \in \liminf_{n\to\infty} Z_{n_k}(x_{n_k})$. Then, there exists a sequence $\{z_{n_k}\}$ in $Z_{n_k}(x_{n_k})$ such that

$$z_{n_k} \to z \text{ as } k \to \infty. \tag{3.7}$$

By (3.3), (3.7), and $f^2_{n_k}$ converges continuously to $f^2$, we obtain that

$$\limsup_{k\to\infty} f^2_{n_k}(x_{n_k}, z_{n_k}) \subseteq f^2(x_0, z) \subseteq \liminf_{k\to\infty} f^2_{n_k}(x_{n_k}, z_{n_k}).$$

Thus, $\lim_{k\to\infty} f^2_{n_k}(x_{n_k}, z_{n_k}) = f^2(x_0, z)$. By (3.5), we obtain that $f^2(x_0, z) \geq 0$. Since $y \in K$ was arbitrary. Then,

$$f(x_0, y) \geq_L 0, \ \forall y \in K.$$

Hence, $x_0 \in S_{LEP}$. So, $\limsup_{n\to\infty} S^n_{LEP} \subseteq S_{LEP}$. This complete the proof. ■

The essentialness of all assumptions are now explained by the following examples.

**Example 3.3.** (Assumption (i) cannot be dropped) Let $X = \mathbb{R}, K = [-1, 1], K_n = [-\frac{1}{n}, \frac{1}{n}]$. Define mapping $f := (f^1, f^2) : K \times K \to \mathbb{R}^2$ by

$$f(x, y) = (y - x, y - x)$$

and sequence mapping $f_n := (f^1_n, f^2_n) : K_n \times K_n \to \mathbb{R}^2$ by

$$f_n(x, y) = ((1 + \frac{1}{n})(y - x), (1 + \frac{1}{n})(y - x)).$$

Obviously, we can check that assumption (ii) be true. From direct computation, we obtain $S_{EP_1(K)} = \{-1\}$ and $S_{EP_1(K_n)} = \{-\frac{1}{n}\}$. Note we also obtain that $Z(x) = \{-1\}$ hence, assumption (iii) is satisfied. Furthermore, we get that $S_{LEP} = \{-1\}$ and $S^n_{LEP} = \{-\frac{1}{n}\}$; however,

$$\{0\} = \limsup_{n\to\infty} S^n_{LEP} \nsubseteq S_{LEP} = \{-1\},$$

i.e., $\{S^n_{LEP}\}$ is not upper convergence in the sense of Painlevé-Kuratowski to $S_{LEP}$.

**Example 3.4.** (Assumption (ii) is essential) Let $X = \mathbb{R}, K = K_n = [1, 2]$. (It is clear that $K_n \xrightarrow{P.K.} K$). Define mapping $f := (f^1, f^2) : K \times K \to \mathbb{R}^2$ by

$$f(x, y) = (0, x - y)$$

and sequence mapping $f_n := (f^1_n, f^2_n) : K_n \times K_n \to \mathbb{R}^2$ by

$$f_n(x, y) = (x - \frac{1}{n}, 0),\ n = 2, 3, 4, \dots.$$

It is clear that $f^i_n$ do not converge to $f^i$ in the sense of Painlevé-Kuratowski. Indeed,

$$x = \limsup_{n\to\infty} f^1_n(x, y) \neq f^1(x, y) = 0,$$

and

$$0 = \limsup_{n\to\infty} f^2_n(x, y) \neq f^2(x, y) = x - y.$$

Thus, it is enough to conclude that the assumption (ii) is not satisfied. From direct computation, we get that $S_{EP_1(K)} = S_{EP_1(K_n)} = [1, 2]$ (we also get $Z(x) = [1, 2]$, and hence the assumption (iii) is satisfied) so, we obtain $S_{LEP} = \{2\}$ and $S^n_{LEP} = [\frac{1}{n}, 2 + \frac{1}{n}]$. At a result, we find that

$$[0, 2] = \limsup_{n\to\infty} S^n_{LEP} \nsubseteq S_{LEP} = \{2\}.$$

Thus, $\{S^n_{LEP}\}$ is not an upper convergence in the sense of Painlevé-Kuratowski to $S_{LEP}$.

**Example 3.5.** (Assumption (iii) is essential) Let $X = \mathbb{R}, K = [0, 1], K_n = [-\frac{1}{n}, 1 + \frac{1}{n}]$. Define mapping $f := (f^1, f^2) : K \times K \to \mathbb{R}^2$ by

$$f(x, y) = (x(x - y)^2, e^y(x - y)),$$

and sequence mapping $f_n := (f^1_n, f^2_n) : K_n \times K_n \to \mathbb{R}^2$ by

$$f_n(x, y) = \left(\left(x + \frac{1}{n}\right)(x - y)^2, \left(1 + \frac{y}{n}\right)^n (x - y)\right).$$

Obviously, we can check that assumption (ii) be true. From direct computation, we obtain $S_{LEP} = (0, 1]$ and $S^n_{LEP} = (-\frac{1}{n}, 1 + \frac{1}{n}]$. Hence, $\limsup_{n\to\infty} S^n_{LEP} \nsubseteq S_{LEP}$. The reason is that assumption (iii) is violated. Indeed, one has $Z(0) = [0, 1]$ and $Z(x) = \{x\}$, for all $x \in (0, 1]$.

All the assumptions in Theorem 3.2, except (iii), are imposed directly on the data of the problem. The following remark includes conditions for assumption (iii) to be fulfilled.

**Remark 3.6.** Since $f^1(x, x) = 0$, for all $x \in X$, one has $x \in Z(x)$. Furthermore, for every $\bar{x} \in K$, $Z$ is lsc at $\bar{x}$ if $Z(\bar{x})$ is singleton; namely, for all $x_n \to \bar{x}$ and $y \in Z(\bar{x})$, we have $y = \bar{x}$. For each $n$, let $y_n = x_n \in Z(x_n)$. It is clear that $y_n \to \bar{x} = y$. In addition, $f^1(x, \cdot)$ is injective, assumption (iii) of Theorem 3.2 is satisfied since $Z(\bar{x})$ is a singleton.

Now, we state our main result to discuss Painlevé-Kuratowski convergence of sequence $\{S^n_{LEP}\}$.

**Theorem 3.7.** *Impose all assumptions of Theorem 3.2 and assume further that*

(iv) *$K$ is convex;*

(v) *$f_1(\cdot, y)$ is strictly concave on $K$, for all $y \in K$.*

*Then $\{S^n_{LEP}\}$ converges to $S_{LEP}$ in the sense of Painlevé-Kuratowski.*

*Proof.* We need only to prove that $S_{LEP} \subset \liminf S^n_{LEP}$. Suppose to the contrary that, there exists $x_0 \in S_{LEP}$ such that $x_0 \notin \liminf S^n_{LEP}$. Then, there is a neighborhood $U_0$ of origin in $X$ such that

$$S^n_{LEP} \cap (x_0 + U_0) = \emptyset, \text{ for } n \text{ sufficiently large.} \tag{3.8}$$

We discuss the following two cases:

**Case 1**. $S_{LEP} = \{x_0\}$. Let $\{x_n\}$ be an arbitrary sequence in $S^n_{LEP}$. Since $K_n \xrightarrow{P.K.} K$, one can assume that $x_n \to \bar{x}_0$ for some $\bar{x}_0 \in K$. The same arguments given in the proof of Theorem 3.2 show that $\bar{x}_0 \in S_{LEP}$. Nothing the fact that $S_{LEP}$ is singleton, we have $x_0 = \bar{x}_0$, and so $x_n \to \bar{x}_0 = x_0$. Thus, $x_n \in x_0 + U_0$ for $n$ large enough. This together with $x_n \in S^n_{LEP}$ implies that $S^n_{LEP} \cap (x_0 + U_0) \neq \emptyset$, for $n$ large enough, which contradicts (3.8).

**Case 2**. There exists $\bar{x} \in S_{LEP}$ satisfying $\bar{x} \neq x_0$. Then, for any $y \in K$, one has

$$f_1(x_0, y) \geq 0 \text{ and } f_1(\bar{x}, y) \geq 0. \tag{3.9}$$

By the strict concavity of $f_1(\cdot, y)$ on $K$, one has

$$f_1(t\bar{x} + (1-t)x_0, y) > 0, \forall t \in (0,1). \tag{3.10}$$

Putting $x(t) := t\bar{x} + (1-t)x_0$ implies that $x(t) \in K$. It is worth noting that for the chosen $U_0$ there exist a neighborhood $U_1$ of origin in $X$ and $t_0 \in (0,1)$ satisfying $U_1 + U_1 \subset U_0$ and $x(t_0) \in x_0 + U_1$. Hence,

$$x(t_0) + U_1 \subset x_0 + U_1 + U_1 \subset x_0 + U_0. \tag{3.11}$$

Since $x(t_0) \in K$ and $K_n \xrightarrow{P.K.} K$, there is $\bar{x}_n \in K_n$ such that $\bar{x}_n \to x(t_0)$. Thus, $\bar{x}_n \in x(t_0) + U_1 \subset x_0 + U_0$, for $n$ sufficiently large. Combining this with (3.8), one has $\bar{x}_n \notin S^n_{LEP}$, for $n$ large enough. Hence, there exists $\bar{y}_n \in K_n$ such that

$$f_1(\bar{x}_n, \bar{y}_n) < 0. \tag{3.12}$$

■

Since $K_n \xrightarrow{P.K.} K$, there is a sequence $\{\bar{y}_n\} \subset K_n$ (taking a subsequence if necessary) such that $\bar{y}_n \to \bar{y}$, for some $\bar{y} \in K$. It follows from (3.12) and the continuous convergence of $f_1$ that

$$f_1(x(t_0), \bar{y}) \leq 0,$$

which contradicts (3.10). The proof is complete. ■

Now, we are in a position to introduce the concept of generalized Hadamard well-posed in the sense of Painlevé-Kuratowski.

**Definition 3.8.** (LEP) is said to be *generalized Hadamard well-posed in the sense of Painlevé-Kuratowski* if its solution set $S_{LEP} \neq \emptyset$ and $x_n \in S^n_{LEP}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \bar{x} \in S_{LEP}$.

**Theorem 3.9.** *Assume that $X$ be a nonempty compact subset of $X$ and all of assumptions in Theorem 3.2 holds. Then* (LEP) *is generalized Hadamard well-posed in the sense of Painlevé-Kuratowski.*

*Proof.* Suppose that the solution set $S_{LEP}$ is nonempty and $x_n \in S^n_{LEP}$. We obtain that $x_n \in K_n$. By the closedness of $S^n_{LEP}$ implies that $S^n_{LEP}$ is compact. Thus, we can find a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that $x_{n_k} \to x_0 \in S^n_{LEP}$ so is $x_0 \in \limsup_{n\to\infty} S^n_{LEP}$. It follows that, $x_0 \in \limsup_{n\to\infty} K_n$. By our assumption, we have $x_0 \in K$. From Theorem 3.2,

$$\limsup_{n\to\infty} S^n_{LEP} \subseteq S_{LEP}.$$

Hence, $x_0 \in S_{LEP}$ and so (LEP) is Hadamard well-posed. This complete the proof. ∎

## 4. Applications

As mentioned in the first section, the lexicographic equilibrium problem (LEP) contains many problems related to optimization with lexicographic cone. Therefore, we can obtain consequences of the results of Section 3 for such special cases. In this section, we only discuss to lexicographic variational inequalities as an example.

Let $K$ be as in the preceding sections, $X$ be a normed space with its dual denoted by $X^*$ and $h_i : X \to X^*$, for $i = 1, 2$. We consider the following lexicographic variational inequality (LVI, for shortly):

(LVI) Find $\bar{x} \in K$ such that

$$(\langle h_1(\bar{x}), y - \bar{x}\rangle, \langle h_2(\bar{x}), y - \bar{x}\rangle) \geq_L 0, \ \forall y \in K.$$

(LVI) can be written in the equivalent way: Find $\bar{x} \in K$ such that

$$\begin{cases} \langle h_1(x), y - x\rangle \geq 0, \forall y \in K, \\ \langle h_2(x), z - x\rangle \geq 0, \forall z \in Z(\bar{x}), \end{cases} \tag{4.1}$$

where

$$Z : K \to 2^X, \ Z(x) := \{y \in K | \, \langle h_1(x), y - x\rangle = 0\}.$$

For each $n \in \mathbb{N}$, let $h_n : K_n \times K_n \to \mathbb{R}^2$. We consider the following sequence of lexicographic variational inequality:

$(\text{LVI})_n$ Find $\bar{x}_n \in K_n$ such that

$$\begin{cases} \langle h^1_n(\bar{x}_n), y - \bar{x}_n\rangle \geq 0, \forall y \in K_n, \\ \langle h^2_n(\bar{x}_n), z - \bar{x}_n\rangle \geq 0, \forall z \in Z_n(\bar{x}_n), \end{cases} \tag{4.2}$$

where

$$Z_n : K_n \to 2^X, \ Z_n(x_n) := \{y_n \in K_n | \, \langle h^1_n(x_n), y_n - x_n\rangle = 0\}.$$

We denoted the solution set of (LVI) and $(\text{LVI})_n$ by $S_{LVI}$ and $S^n_{LVI}$, respectively. By the way, we always assume that $S_{LVI}$ and $S^n_{LVI}$ are nonempty. To convert (LVI) to a special case of (LEP), setting $f_i(x,y) = \langle h_i(x), y-x\rangle$ for $i = 1,2$. The following results are derived from Theorem 3.2 and 3.7.

**Theorem 4.1.** *For (LVI), assume the following conditions are hold:*

(i) $K_n \xrightarrow{P.K.} K$;
(ii) $h_n^i$ *converges continuously to* $h^i$ *for* $i = 1,2$;
(iii) $h^1$ *is surjective.*

*Then,*

$$\limsup_{n\to\infty} S^n_{LVI} \subset S_{LVI}.$$

*Proof.* This is a direct consequence of Theorem 3.2. ∎

**Theorem 4.2.** *For (LVI), impose the assumptions of Theorem 4.1 and the additional conditions:*

(iv) $K$ *is convex*;
(v) $h^1$ *is strictly concave on* $K$.

*Then,* $\{S^n_{LVI}\}$ *converges to* $S_{LVI}$ *in the sense of Painlevé-Kuratowski.*

*Proof.* In order to apply Theorem 3.7, let $f_i(x,y) = \langle h_i(x,y), y-x\rangle$ for $i = 1,2$. It is clear that assumptions (i)-(iv) of Theorem 3.7 are fulfilled. The strictly concave of $h^1(\cdot)$ implies the strictly concave of $f_1(\cdot,y)$, i.e., condition (v) of Theorem 3.7 also holds, and hence Theorem 4.2 is derived from Theorem 3.7 ∎

## 5. Conclusions

In this paper, we study Painlevé-Kuratowski convergence of the solution sets with a sequence of mappings converges continuously. By considering the solution sets of lexicographic vector equilibrium problems, we establish necessary and/or sufcient conditions to be Hadamard well-posed for the mentioned problems in the sense of Painlevé-Kuratowski. Numerous examples are provided to explain that all the imposed assumptions to some results are very relaxed and cannot be dropped. The results in this paper unified, generalized and extended some known results in the literature.

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