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TYKHONOV WELL-POSEDNESS FOR PARAMETRIC GENERALIZED VECTOR EQUILIBRIUM PROBLEMS

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Abstract In this paper, we introduce and analyze the notion of Tykhonov well-posedness and Tykhonov well-posedness in the general sense for parametric generalized vector equilibrium problems. Metric characterizations of well-posedness and well-posedness in the general sense are explained in terms of approximate solution sets. We consider characterizations of these well-posedness under compactness assumptions. Sufficient conditions of well-posedness in the general sense, in the form of boundedness of approximate solution sets, are investigated. Numerous examples are provided to ensure the importance of the imposed assumptions.

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1. Introduction

Equilibrium problems were proposed by Blum and Oettli [1] as a generalization of variational inequalities and optimization problems and include also many optimization related problems like the fixed point and coincidence point problems, the complementarity problems, the traffic equilibria problems, and the Nash equilibrium problems. Recently, the vector equilibrium problem has received much attention by many authors because it provides a unified model including vector optimization problems, vector variational inequality problems, vector complementarity problems and vector saddle point problems as special cases. A great deal of results of various kinds of vector equilibrium problems have been obtained, such as existence or stability of solutions for example, [2–9] and the references therein.

On the other hand, well-posedness plays an important role in the stability analysis and numerical methods for optimization theory and applications. Since any algorithm can generate only an approximating solution sequence which is meaningful only if the

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problem is well-posed under consideration. This fact of well-posedness has inspired many authors to study the well-posedness of extension problems for optimization. Tykhonov [10] introduced well-posedness, which requires the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Well-posedness properties have been intensively studied and the two classical well-posedness notions have been extended and blended. For parametric problems, well-posedness is closely related to stability. Up to now, there have been many works dealing with well-posedness of optimization-related problems as mathematical programming [11, 12], constrained minimization [13–16] variational inequalities [13, 17–20], Nash equilibria [20, 21], and equilibrium problems [14, 22, 23]. Recently, Y.P. Fang considered well-posedness for equilibrium problems and for optimization problems with equilibrium constraints [24]. Very recently, P. Boonman et al. in [25] studied Levitin-Polyak wellposedness and Levitin-Polyak well-posedness in the generalized sense for strong vector mixed quasivariational inequality problems of the Minty type and the Stampacchia type and P.T. Vui et al. in [26] investigated B-well-posedness for set optimization problems involving three kinds of set order relations.

Very Recently, Yu Han and Nan-jing Huang in [27] introduced and investigated parametric generalized vector equilibrium problems. They also had been studied existence, concerning the strong efficient solutions and the weakly efficient solutions, and stability of solutions for a class of generalized vector equilibrium problems.

Motivated by the above works, we study the well-posedness and the well-posedness generalized sense aspects for parametric generalized vector equilibrium problems with set-valued mapping.

The rest of the paper is organized as follows: In section 2, we present some necessary notations, definitions, and lemmas. In section 3, we investigate the well-posedness of the parametric generalized vector equilibrium problems and characterizations of this well-posed. In section 4, we give the notion of well-posedness in the generalized sense by using the well-posedness introduced in the previous section of parametric generalized vector equilibrium problems and consider some sufficient conditions for this problem. Many examples are shown to explain that all imposed assumptions are very relaxed and cannot be dropped.

2. Preliminaries

In this section, we recall some definitions and preliminary results which are used in the next sections. Let X be a Banach space, A be a nonempty closed convex subspace of X, and W, Y and Λ be normed vector spaces. Let $F : X \times X \times W \rightrightarrows Y$ and $K : \Lambda \rightrightarrows X$ be two set-valued mapping. For $(u, \lambda) \in W \times \Lambda$, consider the following parametric generalized vector equilibrium problems consist of finding $x_0 \in K(\lambda) \cap A$ such that

$$(PGVEP) \qquad F(x_0, y, u) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda),$$

where $\Omega \cup \{0\}$ is a cone in Y.

Definition 2.1. Let $F : X \rightrightarrows Y$ be a set-valued map. Then,

(i) F is said to be upper semicontinuous (u.s.c.), if and only if for each closed set $B \subset Y$,
$$\overline{F}(B) := \{x \in X : F(x) \cap B \neq \varnothing\} \text{ is closed in } X.$$

(ii) F is said to be lower semicontinuous (l.s.c.), if and only if for each open set $B \subset Y$,
$$\overline{F}(B) := \{x \in X : F(x) \cap B \neq \varnothing\} \text{ is open in } X.$$

(iii) F is said to be closed if and only if the set $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is closed in $X \times Y$.

Lemma 2.2 ([28]). *A set-valued mapping $F : X \rightrightarrows Y$ is said to be*

(i) *u.s.c. if and only if for any sequence $\{x_n\} \subseteq X$ converging to $x \in X$ and for each sequence $\{y_n\}$ with $y_n \in F(x_n)$, there is $y \in F(x)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to y, where F is compact.*

(ii) *l.s.c. if and only if for every sequence $\{x_n\} \subseteq X$ converging to $x \in X$ and each $y \in F(x)$, there exists a sequence $\{y_n\}$ converging to y with $y_n \in F(x_n)$, for any n.*

(iii) *closed at $x \in X$ if for every sequence $\{x_n\}$ in X converging to x and $\{y_n\}$ converging to y in Y such that $y_n \in F(x_n)$, we have $y \in F(x)$.*

3. Tykhonov Well-Posedness for Parametric Generalized Vector Equilibrium Problems

In this section, we consider the well-posedness of the parametric generalized vector equilibrium problems.

Definition 3.1. Let $\{(u_n, \lambda_n)\} \subseteq W \times \Lambda$ be a sequence converging to $(u, \lambda) \in W \times \Lambda$. A sequence $\{x_n\} \subseteq K(\lambda_n) \cap A$ is said to be an approximating sequence corresponding to $\{(u_n, \lambda_n)\}$ for problem $(PGVEP)$, if there exists a sequence of positive real numbers $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that
$$(F(x_n, y, u_n) + \varepsilon_n e) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda_n).$$

Definition 3.2. The problem $(PGVEP)$ is said to be well-posed if

(i) there exists a unique solution for problem $(PGVEP)$,

(ii) for any sequence $\{(u_n, \lambda_n)\} \subseteq W \times \Lambda$ converging to $(u, \lambda) \in W \times \Lambda$, every approximating sequence $\{x_n\}$ for problem $(PGVEP)$ corresponding to $\{(u_n, \lambda_n)\}$ converges to the unique solution for $(PGVEP)$.

For any $(u, \lambda) \in W \times \Lambda$, we denote the solution of the problem $(PGVEP)$ by $S_F(u, \lambda)$.
$$S_F(u, \lambda) = \{x \in K(\lambda) \cap A : F(x, y, u) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda)\}.$$

In this section, we investigate and characterize well-posedness for problem $(PGVEP)$ via the notion of some approximate solution of problem $(PGVEP)$. In order to consider, let $\delta, \gamma, \varepsilon \in \mathbb{R}_+$ be given and let us introduce the approximate solution set
$$\Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon) := \bigcup_{\substack{\lambda \in B(\bar{\lambda}, \delta) \cap \Lambda \\ u \in B(\bar{u}, \gamma) \cap W}} \{x \in K(\lambda) \cap A : (F(x, y, u) + \varepsilon e) \cap (-\Omega) = \emptyset, \forall y \in K(\lambda)\},$$
where $B(\bar{\lambda}, \delta) = \{\lambda \in \Lambda : \|\lambda - \bar{\lambda}\| \leq \delta\}$ and $B(\bar{u}, \gamma) = \{u \in W : \|u - \bar{u}\| \leq \gamma\}$.

Observe that

$$S_F(\bar{\lambda},\bar{u}) = \Pi(\bar{\lambda},\bar{u},0,0,0) \quad \text{and} \quad \Pi(\bar{\lambda},\bar{u},0,0,0) \subseteq \Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon), \quad \forall \delta,\gamma,\varepsilon > 0.$$

We recall that the diameter of a nonempty set B in Y is defined as

$$\text{diam } B = \sup_{u,v \in B} \|u - v\|.$$

Theorem 3.3. *If the following conditions hold:*

(i) *K is closed and l.s.c. at $\{\bar{\lambda}\}$,*
(ii) *F is l.s.c. on $X \times X \times \{\bar{u}\}$,*
(iii) *$-\Omega$ is open,*

then $(PGVEP)$ is well-posed if and only if $\Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon) \neq \emptyset$, $\delta,\gamma,\varepsilon > 0$ and

$$\text{diam } \Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon) \to 0 \quad \text{as } (\delta,\gamma,\varepsilon) \to (0,0,0).$$

Proof. If the problem $(PGVEP)$ is well-posed, then there is a unique solution x_0. That is

$$x_0 \in S_F(\bar{u},\bar{\lambda}) = \Pi(\bar{\lambda},\bar{u},0,0,0)$$

and hence

$$\Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon) \neq \emptyset, \ \forall \delta,\gamma,\varepsilon > 0 \text{ as } \Pi(\bar{\lambda},\bar{u},0,0,0) \subseteq \Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon).$$

Suppose *diam* $\Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon) \nrightarrow 0$ as $(\delta,\gamma,\varepsilon) \to (0,0,0)$. Then $x_n, x'_n \in \Pi(\bar{\lambda},\bar{u},\delta_n,\gamma_n,\varepsilon_n)$ and there exists $r > 0$, a positive interger m, $\delta_n,\gamma_n,\varepsilon_n > 0$, $(\delta_n,\gamma_n,\varepsilon_n) \to (0,0,0)$ such that

$$\|x_n - x'_n\| > r, \quad \forall n \geq m. \tag{3.1}$$

Since $x_n, x'_n \in \Pi(\bar{\lambda},\bar{u},\delta_n,\gamma_n,\varepsilon_n)$, we get that $\{x_n\} \subseteq K(\lambda_n) \cap A$, $\{x'_n\} \subseteq K(\lambda'_n) \cap A$ and there exist $\lambda_n, \lambda'_n \in B(\bar{\lambda},\delta_n)$, $u_n, u'_n \in B(\bar{u},\gamma_n)$ such that

$$(F(x_n,y,u_n) + \varepsilon_n e) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda_n),$$

and

$$(F(x'_n,y,u'_n) + \varepsilon_n e) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda'_n).$$

As $\delta_n \to 0$ and $\gamma_n \to 0$, we obtain the sequences $\lambda_n \to \bar{\lambda}$, $\lambda'_n \to \bar{\lambda}$, $u_n \to \bar{u}$ and $u'_n \to \bar{u}$. It follows that both $\{x_n\}$ and $\{x'_n\}$ are approximating sequences for $(PGVEP)$. Because $(PGVEP)$ is well-posed, we have both sequences converge to the unique solution x_0 which leads to a contradiction with (3.1).

Conversely, let $\{(u_n,\lambda_n)\}$ be a sequence converging to $(\bar{u},\bar{\lambda})$ and $\{x_n\}$ be approximating sequence for the problem $(PGVEP)$ corresponding to $\{(u_n,\lambda_n)\}$. Then $\{x_n\} \subseteq K(\lambda_n) \cap A$ for some sequences $u_n \to \bar{u}$, $\lambda_n \to \bar{\lambda}$, $\varepsilon_n \to 0$ such that

$$(F(x_n,y,u_n) + \varepsilon_n e) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda_n). \tag{3.2}$$

If we select $\delta_n = \|\lambda_n - \bar{\lambda}\|$ and $\gamma_n = \|u_n - \bar{u}\|$, then $\delta_n \to 0$, $\gamma_n \to 0$ implies that $x_n \in \Pi(\bar{\lambda},\bar{u},\delta_n,\gamma_n,\varepsilon_n)$. As A is closed subspace in Banach space, we obtain that A is complete. Because diam $\Pi(\bar{\lambda},\bar{u},\delta_n,\gamma_n,\varepsilon_n) \to 0$ as $(\delta_n,\gamma_n,\varepsilon_n) \to (0,0,0)$ it implies that $\{x_n\}$ is a Cauchy sequence in A which converges to some $x_0 \in A$. Since K is a closed map on $\{\bar{\lambda}\}$, we have $x_0 \in K(\bar{\lambda}) \cap A$. On the contrary, suppose $x_0 \notin S_F(\bar{u},\bar{\lambda})$, that is, for $(\bar{u},\bar{\lambda}) \in W \times \Lambda$, there exists $y \in K(\bar{\lambda})$, such that

$$F(x_0,y,\bar{u}) \cap (-\Omega) \neq \emptyset,$$

it follows that there exists $z \in F(x_0, y, \bar{u})$ and $z \in -\Omega$. Since K is l.s.c. at $\{\bar{\lambda}\}$, there exists $\{y_n\} \subseteq K(\lambda_n)$ and $y_n \to y$. From (3.2), we obtain that

$$(F(x_n, y_n, u_n) + \varepsilon_n e) \cap (-\Omega) = \emptyset. \tag{3.3}$$

Because F is l.s.c. at $(x_0, y, \bar{u})$ and $(x_n, y_n, u_n) \to (x_0, y, \bar{u})$ with $z \in F(x_0, y, \bar{u})$, there is a sequence $z_n \in F(x_n, y_n, u_n)$ such that $z_n \to z$. Considering $z_n \in F(x_n, y_n, u_n)$ and (3.3), we obtain that

$$z_n + \varepsilon_n e \notin -\Omega.$$

As $Y \backslash -\Omega$ is closed and $z_n + \varepsilon_n e \in Y \backslash -\Omega$ with $z_n \to z$, we get that $z \in Y \backslash -\Omega$. Hence $z \notin -\Omega$, which leads to a contradiction with $z \in -\Omega$. Therefore, $x_0 \in S_F(\bar{u}, \bar{\lambda})$. ■

Corollary 3.4. *If the conditions (i), (ii) and (iii) of the previous theorem hold then (PGVEP) is well-posed if and only if* $S_F(\bar{u}, \bar{\lambda}) \neq \emptyset$ *and*

$$diam\ \Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon) \to 0 \quad as \quad (\delta, \gamma, \varepsilon) \to (0, 0, 0).$$

We now give an example of the metric characterization of Tykhonov well-posedness.

Example 3.5. Let $W = X = Y = \Lambda = \mathbb{R}$, and for all $x, y, u, \lambda \in \mathbb{R}$, $A = (-1, +\infty), \Omega = (0, +\infty)$, and $e = 1$. Define set-valued maps $K : \Lambda \rightrightarrows \mathbb{R}$ and $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$ as follows:

$$K(\lambda) = [0, +\infty)$$

and

$$F(x, y, u) = [y - x, +\infty).$$

$$\begin{aligned} S_F(u, \lambda) &= \{x \in K(\lambda) \cap A : F(x, y, u) \cap -\Omega = \emptyset,\ \forall y \in K(\lambda)\} \\ &= \{x \in [0, +\infty) \cap (-1, +\infty) : [y - x, +\infty) \cap (-(0, \infty)) = \emptyset,\ \forall y \in [0, +\infty)\} \\ &= \{x \in [0, +\infty) : [y - x, +\infty) \cap (-\infty, 0) = \emptyset,\ \forall y \in [0, +\infty)\} \\ &= \{0\}. \end{aligned}$$

$$\begin{aligned} \Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon) &= \bigcup_{\substack{\lambda \in B(\bar{\lambda}, \delta) \cap \Lambda \\ u \in B(\bar{u}, \gamma) \cap W}} \{x \in K(\lambda) \cap A : (F(x, y, u) + \varepsilon e) \cap (-\Omega) = \emptyset, \forall y \in K(\lambda)\} \\ &= \bigcup_{\substack{\lambda \in [\bar{\lambda} - \delta, \bar{\lambda} + \delta] \\ u \in [\bar{u} - \gamma, \bar{u} + \gamma]}} \{x \in [0, +\infty) : [y - x + \varepsilon, +\infty) \cap (-\infty, 0) = \emptyset, \forall y \in [-1, 2]\} \\ &= [0, \varepsilon] \end{aligned}$$

Thus diam $\Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon) = \sup\limits_{u, v \in \Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon)} \|u - v\| = \varepsilon \to 0$ as $(\delta, \gamma, \varepsilon) \to (0, 0, 0)$. By the theorem 3.3, this problem is well-posed.

The following theorem explains that if we give a compact subset A in Y, then the well-posedness of $(PGVEP)$ is equivalent to the existence and uniqueness of solution.

Theorem 3.6. *If A is a compact subset in Y, and the conditions (i), (ii) and (iii) of the theorem 3.3 hold, then (PGVEP) is well-posed if and only if it has a unique solution.*

Proof. Suppose that $(PGVEP)$ is well-posed. By definition of well-posedness, there exists a unique solution for problem $(PGVEP)$.

Conversely, let the unique solution of problem $(PGVEP)$ be $\bar{x}$, then $S_F(\bar{u},\bar{\lambda}) = \{\bar{x}\}$. Let $\{(u_n,\lambda_n)\}$ be a sequence converging to $(\bar{u},\bar{\lambda})$ and $\{x_n\}$ be approximating sequence with respect to $\{(u_n,\lambda_n)\}$. Then there exists a sequenc $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$(F(x_n,y,u_n) + \varepsilon_n e) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda_n). \tag{3.4}$$

As $\{x_n\} \subseteq K(\lambda_n) \cap A$, we obtain $\{x_n\} \subseteq K(\lambda_n)$ and $\{x_n\} \subseteq A$. Since A is a compact subset in Y, the sequence $\{x_n\} \subseteq A$ has a subsequence $\{x_{n_k}\} \subseteq A$ which converges to an element $x_0 \in A$. Furthermore, we get that $\{x_{n_k}\} \subseteq K(\lambda_{n_k})$ with $\lambda_{n_k} \to \bar{\lambda}$ and since K is closed at $\bar{\lambda}$, it follows that $x_0 \in K(\bar{\lambda})$. By the converse part of theorem 3.3, we obtain that $x_0 \in S_F(\bar{u},\bar{\lambda})$. Because problem $(PGVEP)$ has a unique solution, it implies that x_0 and $\bar{x}$ are the same. By the uniqueness of the solution, for every subsequence converges to the unique point x_0. Hence the sequence $\{x_n\}$ converges to x_0 and the problem $(PGVEP)$ is well-posed. ∎

The following example demonstrates the above theorem.

Example 3.7. Let $W = X = Y = \Lambda = \mathbb{R}$, and for all $x,y,u,\lambda \in \mathbb{R}$, $A = [1,3], \Omega = (0,+\infty)$, and $e = 1$. Define set-valued maps $K : \Lambda \rightrightarrows \mathbb{R}$ and $F : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \rightrightarrows \mathbb{R}$ as follows:

$$K(\lambda) = [1,+\infty]$$

and

$$F(x,y,u) = [y - x^2, +\infty).$$

$$\begin{aligned} S_F(\bar{u},\bar{\lambda}) &= \{x \in K(\lambda)\cap A : F(x,y,u)\cap(-\Omega) = \emptyset,\ \forall y \in K(\lambda)\} \\ &= \{x \in [1,+\infty)\cap[1,3] : [y-x^2,+\infty)\cap(-(0,+\infty)) = \emptyset,\ \forall y \in [1,+\infty)\} \\ &= \{x \in [1,3] : [y-x^2,+\infty)\cap(-\infty,0) = \emptyset,\ \forall y \in [1,+\infty)\} \\ &= \{1\}. \end{aligned}$$

Thus the problem has a unique solution. Therefore, by the theorem 3.6, this problem is well-posed.

4. Tykhonov well-posedness in the generalized sense for Parametric Generalized Vector Equilibrium Problems

In this section, we give the notion of well-posedness in the generalized sense by using the well-posedness introduced in the previous section of parametric generalized vector equilibrium problems having more than one solution.

Definition 4.1. The problem $(PGVEP)$ is said to be well-posed in the general sense if

(i) there exist solutions for Problem $(PGVEP)$;

(ii) for any sequence $\{(u_n,\lambda_n)\} \subseteq W \times \Lambda$ converging to $(u,\lambda) \in W\times\Lambda$, every approximating sequence $\{x_n\}$ for problem $(PGVEP)$ corresponding to $\{(u_n,\lambda_n)\}$ has a subsequence converging to element in $S_F(\lambda,u)$.

Obviously, if problem $(PGVEP)$ is well-posed, then it is well-posed in the general sense.

Proposition 4.2. *Well-posedness in the general sense implies that the solution set $S_F(\bar{u},\bar{\lambda})$ of the problem $(PGVEP)$ is nonempty compact set.*

Proof. Let $\{x_n\}$ be any sequence in $S_F(\bar{u},\bar{\lambda})$. Then $x_n \in \Pi(\bar{\lambda},\bar{u},0,0,0)$, we have for any $\delta,\gamma,\varepsilon > 0$ and $x_n \in \Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon)$ since $\Pi(\bar{\lambda},\bar{u},0,0,0) \subseteq \Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon)$. If we select $\lambda_n = \lambda,\ u_n = u,\ \delta_n = \frac{1}{n},\ \gamma_n = \frac{1}{n},\ \varepsilon_n = \frac{1}{n}$ for every n, then x_n is an approximating sequence with respect to $\{(u_n,\lambda_n)\}$. As the problem $(PGVEP)$ is well-posed in the general sense, there exists a subsequence converging to some element in $S_F(\bar{u},\bar{\lambda})$. Therefore, $S_F(\bar{u},\bar{\lambda})$ is compact set. ∎

Theorem 4.3. *The problem $(PGVEP)$ is well-posed in the general sense if and only if $S_F(\bar{u},\bar{\lambda})$ is nonempty compact set and for any approximating sequence $\{x_n\}$ such that*

$$d(x_n, S_F(\bar{u},\bar{\lambda})) \to 0.$$

Proof. Suppose that $(PGVEP)$ is well-posed in the general sense. Then $S_F(\bar{u},\bar{\lambda})$ is nonempty set. Let $\{x_n\}$ be an approximating sequence with respect to $\{(u_n,\lambda_n)\}$. Then there exists $x_0 \in S_F(\bar{u},\bar{\lambda})$ such that $x_n \to x_0$. Thus

$$d(x_n, S_F(\bar{u},\bar{\lambda})) = \inf_{x\in S_F(\bar{u},\bar{\lambda})} \|x_n - x\| \to 0.$$

Conversely, let $\{x_n\}$ be approximating sequence with respect to $\{(u_n,\lambda_n)\}$ and $S_F(\bar{u},\bar{\lambda})$ be a compact. Assume that $d(x_n, S_F(\bar{u},\bar{\lambda})) \to 0$, we obtain $x_n \in S_F(\bar{u},\bar{\lambda})$. Since $S_F(\bar{u},\bar{\lambda})$ is compact set, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converging to some point $x_0 \in S_F(\bar{u},\bar{\lambda})$. Therefore, the problem $(PGVEP)$ is well-posed in the general sense. ∎

Now, we present a metric characterization for the well-posedness in the general sense for $(PGVEP)$. We recall that for two nonempty subset A and B of Y, the distance of a point a from the set B is defined as $d(a,B) = \inf_{b\in B}\|a-b\|$, and the excess of A over B is defined as $e(A,B) = \sup_{a\in A} d(a,B)$.

Theorem 4.4. *The problem $(PGVEP)$ is well-posed in the general sense if and only if $S_F(\bar{u},\bar{\lambda}) \neq \emptyset$, $S_F(\bar{u},\bar{\lambda})$ is compact and*

$$e(\Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon), S_F(\bar{u},\bar{\lambda})) \to 0 \quad as \quad (\delta,\gamma,\varepsilon) \to (0,0,0).$$

Proof. If $(PGVEP)$ is well-posed in the general sense, then $S_F(\bar{u},\bar{\lambda}) \neq \emptyset$. From proporsition 4.2, we have $S_F(\bar{u},\bar{\lambda})$ is compact. Suppose that $e(\Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon), S_F(\bar{u},\bar{\lambda})) \nrightarrow 0$ as $(\delta,\gamma,\varepsilon) \to (0,0,0)$, there exist $r > 0,\ \delta_n,\ \gamma_n,\ \varepsilon_n$ with $\delta_n \to 0,\ \varepsilon_n \to 0,\ \gamma_n \to 0,\ x_n \in \Pi(\bar{\lambda},\bar{u},\delta_n,\gamma_n,\varepsilon_n)$ and a positive integer m such that

$$x_n \notin S_F(\bar{u},\bar{\lambda}) + B(0,r) \quad \forall n \geq m. \tag{4.1}$$

Since $x_n \in \Pi(\bar{\lambda},\bar{u},\delta_n,\gamma_n,\varepsilon_n)$, we get $\{x_n\} \subseteq K(\lambda_n)\cap A$ and there exist $\lambda_n \in B(\bar{\lambda},\delta_n),\ u_n \in B(\bar{u},\gamma_n)$ such that

$$(F(x_n,y,u_n) + \varepsilon_n e) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda_n).$$

As $\delta_n \to 0$ and $\gamma_n \to 0$, we have $\lambda_n \to \bar{\lambda}$ and $u_n \to \bar{u}$. Hence $\{x_n\}$ is approximating sequence for $(PGVEP)$. Since $(PGVEP)$ is well-posed in the general sense, we get that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point in $S_F(\bar{u},\bar{\lambda})$ which is contradiction with (4.1).

Conversely, suppose that $e(\Pi(\bar{\lambda},\bar{u},\delta,\gamma,\varepsilon), S_F(\bar{u},\bar{\lambda})) \to 0$ *as* $(\delta,\gamma,\varepsilon) \to (0,0,0)$, and let $\{x_n\}$ be an approximating sequence for $(PGVEP)$. Then $\{x_n\} \subseteq K(\lambda_n) \cap A$ and there exists $\lambda_n \to \bar{\lambda}$, $u_n \to \bar{u}$ such that

$$(F(x_n, y, u_n) + \varepsilon_n e) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda_n).$$

If we choose $\delta_n = \|\lambda_n - \bar{\lambda}\|$ and $\gamma_n = \|u_n - \bar{u}\|$, then $\delta_n \to 0$, $\gamma_n \to 0$ and $x_n \in \Pi(\bar{\lambda}, \bar{u}, \delta_n, \gamma_n, \varepsilon_n)$. As $e(\Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon), S_F(\bar{u}, \bar{\lambda})) \to 0$ *as* $(\delta, \gamma, \varepsilon) \to (0,0,0)$, we obtain that

$$\begin{aligned} d(x_n, S_F(\bar{u}, \bar{\lambda})) &= \inf_{x \in S_F(\bar{u}, \bar{\lambda})} \|x_n - x\| \le \sup_{x_n \in \Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon)} \inf_{x \in S_F(\bar{u}, \bar{\lambda})} \|x_n - x\| \\ &= e(\Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon), S_F(\bar{u}, \bar{\lambda})) \to 0. \end{aligned}$$

Since $S_F(\bar{u}, \bar{\lambda})$ is compact, there exists $x'_n \in S_F(\bar{u}, \bar{\lambda})$ such that

$$\|x_n - x'_n\| = d(x_n, S_F(\bar{u}, \bar{\lambda})) \to 0. \tag{4.2}$$

By Theorem 4.3, we can conclude that $(PGVEP)$ is well-posed in the general sense. ∎

The following example demonstrates the above theorem.

Example 4.5. Let $W = X = Y = \Lambda = \mathbb{R}$, and for all $x, y, u, \lambda \in \mathbb{R}$, $A = [-1, 3], \Omega = (0, +\infty)$, and $e = 1$. Define set-valued maps $K : \Lambda \rightrightarrows \mathbb{R}$ and $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$ as follows:

$$K(\lambda) = [-1, 2]$$

and

$$F(x, y, u) = [x^2 - y, +\infty).$$

$$\begin{aligned} S_F(\bar{u}, \bar{\lambda}) &= \{x \in K(\lambda) \cap A : F(x, y, u) \cap (-\Omega) = \emptyset, \ \forall y \in K(\lambda)\} \\ &= \{x \in [-1, 2] \cap [-1, 3] : [x^2 - y, +\infty) \cap (-(0, +\infty)) = \emptyset, \ \forall y \in [-1, 2]\} \\ &= \{x \in [-1, 2] : [x^2 - y, +\infty) \cap (-\infty, 0) = \emptyset, \ \forall y \in [-1, 2]\} \\ &= [\sqrt{2}, 2]. \end{aligned}$$

$$\begin{aligned} \Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon) &= \bigcup_{\substack{\lambda \in B(\bar{\lambda}, \delta) \cap \Lambda \\ u \in B(\bar{u}, \gamma) \cap W}} \{x \in K(\lambda) \cap A : (F(x, y, u) + \varepsilon e) \cap (-\Omega) = \emptyset, \forall y \in K(\lambda)\} \\ &= \bigcup_{\substack{\lambda \in [\bar{\lambda} - \delta, \bar{\lambda} + \delta] \\ u \in [\bar{u} - \gamma, \bar{u} + \gamma]}} \{x \in [-1, 2] : [x^2 - y + \varepsilon, +\infty) \cap (-\infty, 0) = \emptyset, \forall y \in [-1, 2]\} \\ &= [\sqrt{2}, \sqrt{2} + \varepsilon] \end{aligned}$$

Hence $h(\Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon), S_F(\bar{u}, \bar{\lambda})) = \sup_{x \in \Pi(\bar{\lambda}, \bar{u}, \delta, \gamma, \varepsilon)} \inf_{y \in S_F(\bar{u}, \bar{\lambda})} \|x - y\| = \sqrt{2} + \varepsilon - \sqrt{2} \to 0$ as $(\delta, \gamma, \varepsilon) \to (0, 0, 0)$. By the theorem 4.4, we obtain that this problem is the general well-posedness.

Theorem 4.6. *If the conditions* $(i), (ii)$ *and* (iii) *of the theorem 3.3 hold, and* A *is a compact set in* Y, *then* $(PGVEP)$ *is well-posed in the general sense if and only if the solution set* $S_F(\bar{u}, \bar{\lambda}) \neq \emptyset$.

Proof. Assume that $(PGVEP)$ is well-posed in the general sense. By the definition of well-posed in the general sense, we have $S_F(\bar{u}, \bar{\lambda}) \neq \emptyset$.

Conversely, let $\{x_n\}$ be an approximating sequence with respect to (u_n, λ_n). Since the converse part of the theorem 3.3, we have $\{x_n\}$ converges to some solution in $S_F(\bar{u}, \bar{\lambda})$. Therefore, the problem $(PGVEP)$ is well-posed in the general sense. ■

The following example explains that boundedness of the set $S_F(\bar{u}, \bar{\lambda})$ of the set $S_F(\bar{u}, \bar{\lambda})$ in theorem cannot be dropped.

Example 4.7. Let $W = X = \Lambda = \mathbb{R}, Y = \mathbb{R}^2$ and for all $x, y, u, \lambda \in \mathbb{R}$, define $A = [-1, \infty)$, $K(\lambda) = \mathbb{R}$, $\Omega = (0, +\infty) \times (0, +\infty)$, $e = (1, 1)$, and $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be defined by

$$F(x, y, u) = [x, +\infty) \times \mathbb{R}_+.$$

$$\begin{aligned} S_F(\bar{u}, \bar{\lambda}) &= \{x \in K(\lambda) \cap A : F(x, y, u) \cap -\Omega = \emptyset\} \\ &= \{x \in \mathbb{R} \cap [-1, \infty) : ([x, +\infty) \times \mathbb{R}_+) \cap (-(0, \infty) \times (0, \infty)) = \emptyset\} \\ &= \{x \in [-1, \infty) : ([x, +\infty) \times \mathbb{R}_+) \cap ((-\infty, 0) \times (-\infty, 0)) = \emptyset\} \\ &= [0, \infty) \end{aligned}$$

If $\{x_n\} = \{n\}, \lambda_n = 1/n,\ u_n = 1/n$, then $x_n \in K(\lambda_n) \cap A = [-1, \infty)$ and hence for any $\varepsilon_n \to 0$, such that

$$(F(x_n, y, u_n) + \varepsilon_n e) \cap -\Omega = \emptyset,$$

$$([x_n, +\infty) \times \mathbb{R}_+ + (\varepsilon_n, \varepsilon_n)) \cap (-(0, +\infty) \times (0, +\infty)) = \emptyset.$$

Therefore, $\{x_n\}$ is approximating sequence but $\{x_n\}$ is not convergent. Thus, this problem is not well-posed.

Theorem 4.8. *If the conditions $(i), (ii)$, and (iii) of the theorem hold, and if for any $x \in X$ there exists some $\varepsilon > 0$ such that $\Pi(\bar{\lambda}, \bar{u}, \varepsilon, \varepsilon, \varepsilon)$ is nonempty and bounded, then $(PGVEP)$ is well-posed in the general sense.*

Proof. Let $\{x_n\}$ be an approximating sequence with respect to $\{(u_n, \lambda_n)\}$ converging to $(\bar{u}, \bar{\lambda})$ for $(PGVEP)$. Then we have $\{x_n\} \subseteq K(\lambda_n) \cap A$ and there exist $\lambda_n \to \bar{\lambda},\ u_n \to \bar{u}$ such that

$$(F(x_n, y, u_n) + \varepsilon_n e) \cap (-\Omega) = \emptyset, \quad \forall y \in K(\lambda_n).$$

If we choose $\delta_n = \|\lambda_n - \bar{\lambda}\|$ and $\gamma_n = \|u_n - \bar{u}\|$, then $\gamma_n \to 0$ and $x_n \in \Pi(\bar{\lambda}, \bar{u}, \delta_n, \gamma_n, \varepsilon_n)$. Let there exist some $\varepsilon > 0$ such that $\Pi(\bar{\lambda}, \bar{u}, \varepsilon, \varepsilon, \varepsilon)$, is nonempty and bounded for each $x \in X$. Then there exists some positive integer m such that $x_n \in \Pi(\bar{\lambda}, \bar{u}, \varepsilon, \varepsilon, \varepsilon)$ for all $n \geq m$. By the boundedness of set $\Pi(\bar{\lambda}, \bar{u}, \varepsilon, \varepsilon, \varepsilon)$, there exists some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$ as $k \to \infty$. Taking the limit of the subsequence $\{x_{n_k}\}$ and the converse part of theorem 3.3, we can conclude that $x_0 \in S_F(\bar{u}, \bar{\lambda})$. ■

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