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WEAK CONVERGENCE FOR EQUILIBRIUM PROBLEMS INVOLVING NONEXPANSIVE AND NONSPREADING MULTIVALUED MAPPINGS

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Abstract In this paper, we introduce iterative methods to approximate a common element of the sets of fixed points of a nonexpansive multivalued mapping and a $\frac{1}{2}$-nonspreading multivalued mapping and the set of solutions of equilibrium problems and prove some weak convergence theorems of the sequences generated by our iterative process under appropriate additional assumptions in Hilbert spaces. As applications, we give the example and numerical results. The results obtained in this paper extend and improve some recent results.

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1. Introduction

In what follows, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty subset of H. The *equilibrium problem* is the problem of finding a point $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0 \tag{1.1}$$

for all $y \in C$. The set of solutions of this problem (1.1) is denoted by $EP(F)$. Given a mapping $A : C \to H$, let $F(x, y) = \langle Ax, y - x \rangle$ for all $y \in C$. Then $\hat{x} \in EP(F)$ if and only if $\hat{x} \subset C$ is a solution of the variational inequality

$$\langle Ax, y - x \rangle \geq 0$$

for all $y \in C$. Since its inception by Blum and Oettli [1] in 1994, the equilibrium problem (1.1) has received much attention due to its applications in a large variety of problems arising in numerous problems in physics, optimizations and economics. Some methods have been proposed to solve the equilibrium problem (see [2–6]).

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In the recent years, solving the problem of finding a common solutions of equilibrium problems and fixed points of a singlevalued mapping in the framework of Hilbert spaces has been intensively studied by many authors (see, for instance, [7–15]).

A subset $C \subset H$ is said to be *proximinal* if, for each $x \in H$,

$$\|x - y\| = d(x, C) = \inf\{\|x - z\| : z \in C\}.$$

Let $CB(C), K(C)$ and $P(C)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal bounded subset of C, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}$$

for $A, B \in CB(C)$, where $d(x, B) = \inf_{z\in B} \|x - z\|$. An element $p \in C$ is called a *fixed point* of $T : C \to CB(C)$ if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$.

Now, we recall that $T : C \to CB(C)$ is:
(1) *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|$$

for all $x, y \in C$;
(2) *quasi-nonexpansive* if

$$H(Tx, Tp) \leq \|x - p\|$$

for all $x \in C$ and $p \in F(T)$.

Recently, the existence of fixed points and the convergence theorems of multivalued mappings have been studied by many authors (see [16–21]).

Hussain and Khan [22] presented the fixed point theorems of a *-nonexpansive multivalued mapping and the strong convergence of its iterates to a fixed point defined on a closed and convex subset of Hilbert spaces by using the best approximation operator $P_T x$, which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$. The convergence theorems and its applications in this direction have been established by many authors (see, for instance, [19, 23, 24]).

In 2011, Song and Cho [25] gave the example for a multivalued mapping T which is not necessary nonexpansive, but P_T is nonexpansive. It would be interesting to study the property of multi-valued mapping T with the help of P_T.

In 2008, Kohsaka and Takahashi [26] introduced a class of mappings which is called *nonspreading mapping*. Let C be a subset of Hilbert spaces H. A mapping $T : C \to C$ is said to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. Recently, in 2009, Iemoto and Takahashi [27] showed that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Ty, y - Ty\rangle$$

for all $x, y \in C$. In 2012, Jiang and Su [28] introduced an iterative scheme for finding a common element of the set of fixed points of a nonexpansive singlevalued mappings and nonspreading singlevalued mappings and the set of solution of an equilibrium problem on the setting of real Hilbert spaces. Recently, in 2013, Eslamian [29] extended the results to a finite family of nonspreading singlevalued mappings and a finite family of nonexpansive multivalued mappings.

Also, recently, in 2013, Liu [9] introduced the following class of multivalued mappings: A mapping $T : C \to CB(C)$ is said to be *nonspreading* if

$$2\|u_x - u_y\|^2 \leq \|u_x - y\|^2 + \|u_y - x\|^2$$

for some $u_x \in Tx$ and $u_y \in Ty$ for all $x, y \in C$. Also, he proved a weak convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points.

In this paper, we introduce, by using Hausdorff metric, the class of nonspreading multivalued mappings. We say that a mapping $T : C \to CB(C)$ is a *k-nonspreading multivalued mapping* if there exists $k > 0$ such that

$$H(Tx, Ty)^2 \leq k\big(d(Tx, y)^2 + d(x, Ty)^2\big) \tag{1.2}$$

for all $x, y \in C$. It is easy to see that, if T is $\frac{1}{2}$-nonspreading, then T is nonspreading in the case of singlevalued mappings (see [26, 30]). Moreover, if T is a $\frac{1}{2}$-nonspreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Indeed, for all $x \in C$ and $p \in F(T)$, we have

$$\begin{aligned} 2H(Tx, Tp)^2 &\leq d(Tx, p)^2 + d(x, Tp)^2 \\ &\leq H(Tx, Tp)^2 + \|x - p\|^2 \end{aligned}$$

and so it follows that

$$H(Tx, Tp) \leq \|x - p\|. \tag{1.3}$$

We now give an example of a $\frac{1}{2}$-nonspreading multivalued mapping which is not nonexpansive.

Example 1.1. Consider $C = [-3, 0]$ with the usual norm. Define a multi-valued mapping $T : C \to CB(C)$ by

$$Tx = \begin{cases} \{0\}, & x \in [-2, 0]; \\ \big[-\frac{|x|}{|x|+1}, 0\big], & x \in [-3, -2). \end{cases}$$

Now, we show that T is $\frac{1}{2}$-nonspreading. In fact, we have the following 3 cases:

Case 1: If $x, y \in [-2, 0]$, then $H(Tx, Tx) = 0$.

Case 2: If $x \in [-2, 0]$ and $y \in [-3, -2)$, then $Tx = \{0\}$ and $Ty = \big[-\frac{|y|}{|y|+1}, 0\big]$. This implies that

$$2H(Tx, Ty)^2 = 2\Big(\frac{|y|}{|y|+1}\Big)^2 < 2 < y^2 \leq d(Tx, y)^2 + d(x, Ty)^2.$$

Case 3: If $x, y \in [-3, -2)$, then $Tx = \big[-\frac{|x|}{|x|+1}, 0\big]$ and $Ty = \big[-\frac{|y|}{|y|+1}, 0\big]$. This implies that

$$2H(Tx, Ty)^2 = 2\Big(\frac{|x|}{|x|+1} - \frac{|y|}{|y|+1}\Big)^2 < 2 < d(Tx, y)^2 + d(x, Ty)^2.$$

On the other hand, T is not nonexpansive since, for $x = -2$ and $y = -\frac{5}{2}$, we have $Tx = \{0\}$ and $Ty = \big[-\frac{5}{7}, 0\big]$. This shows that

$$H(Tx, Ty) = \frac{5}{7} > \frac{1}{2} = \big|-2 - \big(-\frac{5}{2}\big)\big| = \|x - y\|.$$

In this paper, inspired by Jiang and Su [28], Eslamian [29] and Liu [9], we study the definition of a nonspreading multivalued mapping by using the Hausdorff metric and introduce an iterative method to approximate a common solution of the equilibrium problem and a common fixed point problem for a $\frac{1}{2}$-nonspreading multivalued mapping and a nonexpansive multivalued mapping. Furthermore, we prove the weak convergence theorem in Hilbert spaces and, also, give some examples and numerical results.

2. Preliminaries

We now provide some basic results for the proof. In a Hilbert space H, we know the following lemma:

Lemma 2.1. *Let H be a real Hilbert space. Then the following equations hold:*
(1) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in H$*;*
(2) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ *for all* $x, y \in H$*;*
(3) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $t \in [0,1]$ *and* $x, y \in H$*;*
(4) *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in* H *which converges weakly to* $z \in H$*, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2$$

for all $y \in H$*.*

A space X is said to satisfy *Opial's condition* if, for any sequence x_n with $x_n \rightharpoonup x$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is known that every Hilbert space satisfies Opial's condition.

Lemma 2.2. [31] *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that* $\lim_{n\to\infty} \|x_n - u\|$ *and* $\lim_{n\to\infty} \|x_n - v\|$ *exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.3. [32] *Let C be a nonempty weakly compact subset of a Banach space X with Opial's condition and $T : C \to K(X)$ be a nonexpansive mapping. Then $I - T$ is demiclosed.*

Lemma 2.4. [1] *Let D be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)-(A4) and let $r > 0$ and $x \in H$. Then there exists $z \in D$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in D$.

Assumptiom 2.5. [33] Let D be a nonempty closed convex subset of a Hilbert space H. Let $F : D \times D \to \mathbb{R}$ be a bifunction satisfying the following assumptions:
(A1) $F(x, x) = 0$ for all $x \in D$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in D$;
(A3) For each $x, y, z \in D$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) For each $x \in D$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 2.6. [2] *Assume that $F : C \times C \to \mathbb{R}$ satisfying (A1)-(A4). For $r > 0$, $x \in H$, define the mapping $T_r : H \to D$ as follows:*

$$T_r(x) = \Big\{z \in D : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in D\Big\}.$$

Then the followings hold:

(1) *T_r is single-value;*

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Condition (A)**.** Let H be a Hilbert space and C be a subset of H. A multivalued mapping $T : C \to CB(C)$ is said to satisfy *Condition* (A) if $\|x - p\| = d(x, Tp)$ for all $x \in H$ and $p \in F(T)$.

Assumptiom 2.7. We see that T satisfies Condition (A) if and only if $Tp = \{p\}$ for all $p \in F(T)$. It is known that the best approximation operator P_T, which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$, also satisfies Condition (A).

Lemma 2.8. [34] *Let C be a closed and convex subset of a real Hilbert space H and $T : C \to K(C)$ be a $\frac{1}{2}$-nonspreading multi-valued mapping. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

3. Main results

We are now ready to prove the weak convergence theorem for $\frac{1}{2}$-nonspreading multivalued mapping and nonexpansive multivalued mapping in Hilbert spaces.

Theorem 3.1. *Let H be a Hilbert space, C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $S : C \to K(C)$ be a $\frac{1}{2}$-nonspreading multivalued mapping and $T : C \to K(C)$ be a nonexpansive multivalued mapping such that $\Theta = F(S) \cap F(T) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \\ x_{n+1} \in (1 - \alpha_n)u_n + \alpha_n\big\{\beta_n S u_n + (1 - \beta_n)T u_n\big\} \end{cases} \tag{3.1}$$

for all $y \in C$ and $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $\subset (0, 1)$ and $r_n \subset (0, \infty)$ satisfying the following conditions: (1) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$*;*

(2) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$*;*

(3) $\liminf_{n\to\infty} r_n > 0$.

If S and T satisfy Condition (A), then the sequences $\{x_n\}$ and $\{u_n\}$ defined by (3.19) *converge weakly to an element of Θ.*

Proof. Let $p \in \Theta$. We divide the proof into five Steps as follows:

Step 1. Show that $\{x_n\}$ is bounded. From $u_n = T_{r_n} x_n$, we have

$$\|u_n - p\| = \|T_{r_n} x_n T_{r_n} p\| \leq \|x_n - p\| \tag{3.2}$$

for all $n \geq 1$. Since T satisfies Condition (A), for all $z_n \in Tu_n$,

$$\begin{aligned} \|(1-\alpha_n)u_n + \alpha_n z_n - p\|^2 &\leq (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\|z_n - p\|^2 \\ &= (1-\alpha_n)\|u_n - p\|^2 + \alpha_n d(z_n, Tp)^2 \\ &\leq (1-\alpha_n)\|u_n - p\|^2 + \alpha_n H(Tu_n, Tp)^2 \\ &\leq \|u_n - p\|^2. \end{aligned} \tag{3.3}$$

Similarly, since S satisfies Condition (A), for all $y_n \in Su_n$,

$$\begin{aligned} \|(1-\alpha_n)u_n + \alpha_n y_n - p\|^2 &\leq (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\|y_n - p\|^2 \\ &= (1-\alpha_n)\|u_n - p\|^2 + \alpha_n d(y_n, Sp)^2 \\ &\leq (1-\alpha_n)\|u_n - p\|^2 + \alpha_n H(Su_n, Sp)^2 \\ &\leq \|u_n - p\|^2. \end{aligned} \tag{3.4}$$

From (3.2), (3.3) and (3.4), for all $y_n \in Su_n$ and $z_n \in Tu_n$, we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\beta_n\{(1-\alpha)u_n + \alpha_n y_n\} + (1-\beta_n)\{(1-\alpha)u_n + \alpha_n z_n\} - p\|^2 \\ &\leq \beta_n\|(1-\alpha)u_n + \alpha_n y_n - p\|^2 + (1-\beta_n)\|(1-\alpha)u_n + \alpha_n z_n - p\|^2 \\ &\leq \beta_n\|(1-\alpha)u_n + \alpha_n y_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2 \\ &\leq \|u_n - p\|^2 \\ &\leq \|x_n - p\|^2 \end{aligned} \tag{3.5}$$

for all $n \geq 1$. Hence $\lim_{n\to\infty}\|x_n - p\|$ exists. Therefore, $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. Show that there exists a subsequence $\{u_{n_i}\} \subset \{u_n\}$ which converges weakly to $q \in F(S) \cap F(T)$. From (3.5), we have

$$\begin{aligned} 0 &\leq \|u_n - p\|^2 - \beta_n\|(1-\alpha)u_n + \alpha_n y_n - p\|^2 - (1-\beta_n)\|u_n - p\|^2 \\ &= \beta_n\big(\|u_n - p\|^2 - \|(1-\alpha)u_n + \alpha_n y_n - p\|^2\big) \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \end{aligned} \tag{3.6}$$

Since $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ and $\lim_{n\to\infty}\|x_n - p\|$ exists, it follows from (3.6) that

$$\lim_{n\to\infty}\big(\|u_n - p\|^2 - \|(1-\alpha)u_n + \alpha_n y_n - p\|^2\big) = 0. \tag{3.7}$$

By Lemma 2.1, we have

$$\begin{aligned} &\|(1-\alpha)u_n + \alpha_n y_n - p\|^2 \\ \leq\ &(1-\alpha)\|u_n - p\|^2 + \alpha_n\|y_n - p\|^2 - (1-\alpha_n)\alpha_n\|u_n - y_n\|^2 \\ =\ &(1-\alpha)\|u_n - p\|^2 + \alpha_n d(y_n, Sp)^2 - (1-\alpha_n)\alpha_n\|u_n - y_n\|^2 \\ \leq\ &(1-\alpha)\|u_n - p\|^2 + \alpha_n H(Su_n, Sp)^2 - (1-\alpha_n)\alpha_n\|u_n - y_n\|^2 \\ \leq\ &\|u_n - p\|^2 - (1-\alpha_n)\alpha_n\|u_n - y_n\|^2. \end{aligned}$$

This implies that

$$(1-\alpha_n)\alpha_n\|u_n - y_n\|^2 \leq \|u_n - p\|^2 - \|(1-\alpha)u_n + \alpha_n y_n - p\|^2. \tag{3.8}$$

Since $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n > 0$, it follows from (3.8) that

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.9}$$

Since $\{u_n\}$ is a bounded sequence, there exists a subsequence $\{u_{n_i}\} \subset \{u_n\}$ such that $\{u_{n_i}\}$ converges weakly to $q \in C$. From Lemma 2.8, we obtain $q \in F(S)$.

Now, we show that $q \in F(T)$. From (3.2), (3.3) and (3.4), it follows that, for all $y_n \in Su_n$ and $z_n \in Tu_n$,

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\beta_n\{(1-\alpha)u_n+\alpha_n y_n\}+(1-\beta_n)\{(1-\alpha)u_n+\alpha_n z_n\}-p\|^2\\ &\le \beta_n\|(1-\alpha)u_n+\alpha_n y_n-p\|^2+(1-\beta_n)\|(1-\alpha)u_n+\alpha_n z_n-p\|^2\\ &\le \beta_n\|u_n-p\|^2+(1-\beta_n)\|(1-\alpha)u_n+\alpha_n z_n-p\|^2\\ &\le \|u_n-p\|^2\\ &\le \|x_n-p\|^2 \end{aligned} \tag{3.10}$$

for all $n \ge 1$. This implies that

$$\begin{aligned} 0 &\le \|u_n-p\|^2-\beta_n\|u_n-p\|^2-(1-\beta_n)\|(1-\alpha)u_n+\alpha_n z_n-p\|^2\\ &= (1-\beta_n)\big(\|u_n-p\|^2-\|(1-\alpha)u_n+\alpha_n z_n-p\|^2\big)\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2. \end{aligned} \tag{3.11}$$

Since $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$ and $\lim_{n\to\infty}\|x_n-p\|$ exists, it follows from (3.11) that

$$\lim_{n\to\infty}\big(\|u_n-p\|^2-\|(1-\alpha)u_n+\alpha_n z_n-p\|^2\big)=0. \tag{3.12}$$

Also, by Lemma 2.1, it follows that

$$\begin{aligned} &\|(1-\alpha)u_n+\alpha_n z_n-p\|^2\\ \le\;& (1-\alpha)\|u_n-p\|^2+\alpha_n\|z_n-p\|^2-(1-\alpha_n)\alpha_n\|u_n-z_n\|^2\\ =\;& (1-\alpha)\|u_n-p\|^2+\alpha_n d(z_n,Tp)^2-(1-\alpha_n)\alpha_n\|u_n-y_n\|^2\\ \le\;& (1-\alpha)\|u_n-p\|^2+\alpha_n H(Tu_n,Tp)^2-(1-\alpha_n)\alpha_n\|u_n-z_n\|^2\\ \le\;& \|u_n-p\|^2-(1-\alpha_n)\alpha_n\|u_n-z_n\|^2, \end{aligned}$$

which implies that

$$(1-\alpha_n)\alpha_n\|u_n-z_n\|^2 \le \|u_n-p\|^2-\|(1-\alpha)u_n+\alpha_n z_n-p\|^2. \tag{3.13}$$

Since $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n>0$, from (3.8), it follows that

$$\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.14}$$

Using (3.9), we obtain

$$\lim_{n\to\infty} d(u_n,Tu_n) \le \lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.15}$$

Thus, by Lemma 2.3, we conclude that $q \in F(T)$.

Step 3. Show that $\|x_n-u_n\|\to 0$ as $n\to\infty$. Since $u_n=T_{r_n}x_n$, we see that

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{r_n}x_n-T_{r_n}p\|^2\\ &\le \langle T_{r_n}x_n-T_{r_n}p, x_n-p\rangle\\ &= \langle u_n-p, x_n-p\rangle\\ &= \frac{1}{2}\big(\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2\big),\end{aligned}$$

which yields

$$\|u_n-p\|^2 \le \|x_n-p\|^2-\|x_n-u_n\|^2.$$

From (3.3), we have

$$\|x_n-u_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists,

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.16}$$

Step 4. Show that $q \in EP(F)$. Since $u_n = T_{r_n}x_n$,

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0$$

for all $y \in C$. From (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}).$$

Since $\liminf_{n\to\infty} r_n > 0$ and $u_{n_i} \rightharpoonup q$, by (A4) and (3.16), we have

$$F(y, q) \leq 0$$

for all $y \in C$. Replacing y with $y_t = ty + (1-t)q$ for any $t \in [0,1]$, from (A1) and (A4), it follows that

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, q) \leq tF(y_t, y)$$

and hence

$$F(ty + (1-t)q, y) \geq 0$$

for any $t \in [0,1]$ and $y \in C$. So, $F(q, y) \geq 0$ for all $y \in C$ by (A3) and letting $t \to 0^+$. Hence $q \in EP(F)$ and so $q \in \Theta$.

Step 5. Show that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of Θ. It is sufficient to show that $\omega_w(x_n)$ is a single point set. Let $p, q \in \omega_w(x_n)$ and $\{x_{n_k}\}, \{x_{n_m}\} \subset \{x_n\}$ be such that $x_{n_k} \rightharpoonup p$ and $x_{n_m} \rightharpoonup q$. From (3.16), we have $u_{n_k} \rightharpoonup p$ and $u_{n_m} \rightharpoonup q$. By Step 2 and Step 4, we have $p, q \in \Theta$. Applying Lemma 2.2, we obtain $p = q$. This completes the proof. ∎

If $Tp = \{p\}$ for all $p \in F(T)$, then T satisfies Condition (A) and so we obtain the following result:

Theorem 3.2. *Let H be a Hilbert space, C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $S : C \to K(C)$ be a $\frac{1}{2}$-nonspreading multi-valued mapping and $T : C \to K(C)$ be a nonexpansive multi-valued mapping such that $\Theta = F(S) \cap F(T) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \\ x_{n+1} \in (1-\alpha_n)u_n + \alpha_n\{\beta_n S u_n + (1-\beta_n)T u_n\} \end{cases} \tag{3.17}$$

for all $y \in C$ and $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $\subset (0,1)$ and $r_n \subset (0,\infty)$ satisfying the following conditions:

(1) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(3) $\liminf_{n\to\infty} r_n > 0$.

If $Sp = Tp = \{p\}$ for all $p \in F(S) \cap F(T)$, then the sequences $\{x_n\}$ and $\{u_n\}$ defined by (3.19) converge weakly to an element of Θ.

Since P_T satisfies Condition (A), we also obtain the following results:

Theorem 3.3. *Let H be a Hilbert space, C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $S, T : C \to P(C)$ be two multi-valued mappings such that $\Theta = F(S) \cap F(T) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \\ x_{n+1} \in (1-\alpha_n)u_n + \alpha_n\{\beta_n P_S u_n + (1-\beta_n)P_T u_n\} \end{cases} \tag{3.18}$$

for all $y \in C$ and $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $\subset (0,1)$ and $r_n \subset (0,\infty)$ satisfying the following conditions:
(1) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(3) $\liminf_{n\to\infty} r_n > 0$.

If P_S is $\frac{1}{2}$-nonspreading and P_T is nonexpansive such that $I-S$ and $I-T$ are demiclosed at 0, then the sequences $\{x_n\}$ and $\{u_n\}$ defined by (3.19) converge weakly to an element of Θ.

Corollary 3.4. *Let H be a Hilbert space, C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $S : C \to C$ be a $\frac{1}{2}$-nonspreading mapping and $T : C \to C$ be a nonexpansive mapping such that $\Theta = F(S) \cap F(T) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n\{\beta_n S u_n + (1-\beta_n)T u_n\} \end{cases} \tag{3.19}$$

for all $y \in C$ and $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $\subset (0,1)$ and $r_n \subset (0,\infty)$ satisfying the following conditions:
(1) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(3) $\liminf_{n\to\infty} r_n > 0$.

Then the sequences $\{x_n\}$ and $\{u_n\}$ defined by (3.19) converge weakly to an element of Θ.

Remark 3.5. (1) Theorem 3.1-3.3 extend the main results of Jiang and Su [28] from a nonspreading single-valued mapping to a nonspreading multi-valued mapping and a nonexpansive single-valued mapping to a nonexpansive multi-valued mapping.
(2) If S is a nonspreading single-valued mapping and T is a nonexpansive single-valued mapping, then we obtain Corollary 3.4.

4. Examples and Numerical Results

In this section, we give examples and numerical results for supporting our main theorem.

Example 4.1. Let $H = \mathbb{R}$ and $C = [-3, 0]$. Let $F(x, y) = (x+3)(y-x)$ for all $x, y \in C$,

$$Sx = \begin{cases} \{-3\}, & x \in [-3, -1]; \\ \big[-3, -2 - \frac{|x|}{|x|+1}\big], & x \in (-1, 0], \end{cases}$$

and

$$Tx = \Big[-3, 3\sin\big(\frac{\pi x}{6}\big)\Big].$$

Choose $\alpha_n = \frac{1}{10n}$, $\beta = \frac{1}{n}$ and $r_n = \frac{n}{n+1}$. It is easy to check that F satisfies all the conditions in Theorem 3.1. For each $r > 0$ and $x \in C$, we first find $u \in C$ such that

$$F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0 \tag{4.1}$$

for all $y \in C$. We see that

$$\begin{aligned} F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0 \quad &\Longleftrightarrow \quad (u+3)(y-u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0 \\ &\Longleftrightarrow \quad r(u+3)(y-u) + (y-u)(u-x) \geq 0 \\ &\Longleftrightarrow \quad (y-u)((1+r)u - (x-3r)) \geq 0. \end{aligned}$$

By Lemma 2.6, we know that $T_r^F x$ is single-valued. Hence $u = \frac{x-3r}{1+r}$.

Next, we find

$$x_{n+1} \in (1-\alpha_n)u_n + \alpha_n\big\{\beta_n S u_n + (1 - \beta_n T u_n)\big\},$$

where $u_n = \frac{x_n - 3r_n}{1+r_n}$. From

$$Sx = \begin{cases} \{-3\}, & x \in [-3, -1]; \\ \big[-3, -2 - \frac{|x|}{|x|+1}\big], & x \in (-1, 0], \end{cases}$$

and

$$Tx = \Big[-3, 3\sin(\frac{\pi x}{6})\Big],$$

we have

$$x_{n+1} = \big(1 - \frac{1}{10n}\big)\left(\frac{x_n - 3\big(\frac{n}{n+1}\big)}{1 + \big(\frac{n}{n+1}\big)}\right) + \frac{1}{10n}\Big\{\frac{1}{n}y_n + \big(1 - \frac{1}{n}\big)z_n\Big\}, \tag{4.2}$$

where

$$y_n \in \begin{cases} \{-3\}, & u_n \in [-3, -1]; \\ \big[-3, -2 - \frac{|u_n|}{|u_n+1|}\big], & u_n \in (-1, 0], \end{cases}$$

and

$$z_n \in \Big[-3, 3\sin(\frac{\pi u_n}{6})\Big].$$

Finally, if we compute the numerical results by choosing $x_1 = 0$ and taking randomly y_n and z_n in the above intervals, then we obtain the following Tables:

n	u_n	y_n	z_n	x_n	$\|x_{n+1} - x_n\|$
1	-1.00000E+00	-3.00000E+00	-1.60462E+00	0.00000E+00	1.20000E+00
2	-1.92000E+00	-3.00000E+00	-2.90109E+00	-1.20000E+00	7.71527E-01
3	-2.41230E+00	-3.00000E+00	-2.95694E+00	-1.97153E+00	4.59407E-01
4	-2.68385E+00	-3.00000E+00	-2.96544E+00	-2.43093E+00	2.60174E-01
5	-2.83151E+00	-3.00000E+00	-2.99025E+00	-2.69111E+00	1.43619E-01
6	-2.91101E+00	-3.00000E+00	-2.99814E+00	-2.83473E+00	7.77371E-02
7	-2.95331E+00	-3.00000E+00	-2.99982E+00	-2.91246E+00	4.15147E-02
8	-2.97564E+00	-3.00000E+00	-2.99993E+00	-2.95398E+00	2.19607E-02
9	-2.98734E+00	-3.00000E+00	-2.99997E+00	-2.97594E+00	1.15374E-02
10	-2.99344E+00	-3.00000E+00	-2.99999E+00	-2.98748E+00	6.02878E-03
⋮	⋮	⋮	⋮	⋮	⋮
21	-3.00000E+00	-3.00000E+00	-3.00000E+00	-2.99999E+00	3.99991E-06

Table 1. Numerical results of Example 4.1 being randomized in the first time.

n	u_n	y_n	z_n	x_n	$\|x_{n+1} - x_n\|$
1	-1.00000E+00	-3.00000E+00	-2.36478E+00	0.00000E+00	1.20000E+00
2	-1.92000E+00	-3.00000E+00	-2.83312E+00	-1.20000E+00	7.69828E-01
3	-2.41133E+00	-3.00000E+00	-2.94717E+00	-1.96983E+00	4.59951E-01
4	-2.68321E+00	-3.00000E+00	-2.98064E+00	-2.42978E+00	2.60989E-01
5	-2.83133E+00	-3.00000E+00	-2.98890E+00	-2.69077E+00	1.43756E-01
6	-2.91090E+00	-3.00000E+00	-2.99898E+00	-2.83452E+00	7.78448E-02
7	-2.95326E+00	-3.00000E+00	-2.99917E+00	-2.91237E+00	4.15524E-02
8	-2.97560E+00	-3.00000E+00	-2.99976E+00	-2.95392E+00	2.19868E-02
9	-2.98732E+00	-3.00000E+00	-2.99996E+00	-2.97591E+00	1.15529E-02
10	-2.99343E+00	-3.00000E+00	-2.99999E+00	-2.98746E+00	6.03691E-03
⋮	⋮	⋮	⋮	⋮	⋮
21	-3.00000E+00	-3.00000E+00	-3.00000E+00	-2.99999E+00	4.00538E-06

Table 2. Numerical results of Example 4.1 being randomized in the second time.

From Table 1 and Table 2, we see that -3 is a solution in Example 4.1.

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