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Metric-preserving Functions, W-distances and Cauchy W-distances

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Abstract : A function $f : [0,\infty) \to [0,\infty)$ is called *metric-preserving* if for every metric space (X,d), $f \circ d$ is a metric on X. The notion of w-distance on a metric space was introduced by Shioji, Suzuki and Takahashi in 1998. By a *w-distance* on a metric space (X,d), they mean a function $p : X \times X \to [0,\infty)$ having the properties that for all $x,y,z \in X, p(x,z) \leq p(x,y) + p(y,z)$, $p(x,\cdot)$ is lower semicontinuous and for any $\varepsilon > 0$, there is a $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$. Then we call such a p a *Cauchy w-distance* if every Cauchy sequence (x_n) in (X,d) has the property relating to p that for every $\varepsilon > 0$, there exists a positive integer N such that $p(x_n,x_m) < \varepsilon$ for all $m > n \geq N$. Our purpose is to show that the metric $f \circ d$ is a w-distance on (X,d) if f is lower semicontinuous and it is a Cauchy w-distance if f is continuous.

Keywords : Metric-preserving function, w-distance, Cauchy w-distance.
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1 Introduction

A function $f : [0,\infty) \to [0,\infty)$ is called *metric-preserving* if for every metric space (X,d), $f \circ d$ is a metric on X.

Three important necessary conditions for $f : [0,\infty) \to [0,\infty)$ to be metric-preserving are as follows:

Proposition 1.1 ([1]). *If $f : [0,\infty) \to [0,\infty)$ is metric-preserving, then*

(C1) *for all* $x \in [0,\infty), f(x) = 0 \Leftrightarrow x = 0.$

Proposition 1.2 ([2]). *If $f : [0,\infty) \to [0,\infty)$ is metric-preserving, then*

(C2) *for all* $x,y \in [0,\infty)$, $f(x+y) \leq f(x) + f(y).$

Proposition 1.3 ([2]). *If $f : [0,\infty) \to [0,\infty)$ is metric-preserving and f is continuous at* 0*, then f is continuous on* $[0,\infty)$.

Example 1.4. Define $f : [0,\infty) \to [0,\infty)$ by

$$f(x) = \begin{cases} x & \text{if } \ 0 \le x \le 1, \\ \frac{1}{2} & \text{if } \ x > 1. \end{cases}$$

By Proposition 1.3, f is not metric-preserving. However, f satisfies (C1) and (C2).

It is known that adding "nondecreasing" to (C1) and (C2) and "concave" to (C1) yield sufficient conditions for f to be metric-preserving.

Proposition 1.5 ([2]). *If $f : [0,\infty) \to [0,\infty)$ satisfies* (C1) *and* (C2) *and f is nondecreasing, then f is metric-preserving.*

Proposition 1.6 ([1]). *If $f : [0,\infty) \to [0,\infty)$ satisfies* (C1) *and f is concave that is, $f(\alpha x + (1-\alpha)y) \ge \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in X$ and $\alpha \in [0,1]$, then f is metric-preserving.*

Example 1.7. (1) Let $a > 1$ and $f(x) = \log_a(1+x)$ for all $x \in [0,\infty)$. Then f satisfies (C1) and f is increasing. Also, for $x, y \in [0,\infty)$,

$$\begin{aligned} f(x+y) \ = \ \log_a(1+x+y) &\le \log_a(1+x+y+xy) \\ &= \log_a((1+x)(1+y)) \\ &= \log_a(1+x) + \log_a(1+y) \\ &= f(x) + f(y). \end{aligned}$$

By Proposition 1.5, f is metric-preserving.

(2) Let $r \in (0,1]$ and $g : [0,\infty) \to [0,\infty)$ defined by $g(x) = x^r$ for all $x \in [0,\infty)$. Then g satisfies (C1) and g is concave. Hence by Proposition 1.6, g is metric-preserving.

(3) Let $c > 0$ and define $h : [0,\infty) \to [0,\infty)$ by

$$h(x) = \begin{cases} 0 & \text{if } \ x = 0, \\ c & \text{if } \ x > 0. \end{cases}$$

Then h is metric-preserving by Proposition 1.5.

The metric-preserving functions f, g and h in Example 1.7 are obtained from [2] or [3].

Notice that the metric-preserving functions f, g and h in Example 1.7 are all nondecreasing. In fact, a metric-preserving function may be strictly decreasing on $(0,\infty)$.

Example 1.8 ([2]). Define $f : [0,\infty) \to [0,\infty)$ by

$$f(x) = \begin{cases} 0 & \text{if } \ x = 0, \\ 1 + \frac{1}{x+1} & \text{if } \ x > 0. \end{cases}$$

Then f is a metric-preserving function which is strictly decreasing and continuous on $(0,\infty)$.

The following necessary condition of a metric-preserving function will be used.

Proposition 1.9 ([2]). *If $f : [0, \infty) \to [0, \infty)$ is metric-preserving, then for each $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\text{for all } x \in [0, \infty),\ f(x) < \delta \Rightarrow x < \varepsilon.$$

Recall that for a topological space X, a function $f : X \to \mathbb{R}$ is said to be *lower semicontinuous* on X if for every $a \in \mathbb{R}$, $f^{-1}((-\infty, a])$ is closed in X, or equivalently, $f^{-1}((a, \infty))$ is open in X. Hence every continuous function from X into $\mathbb{R}$ is lower semicontinuous.

For nonempty sets X and Y, if $f : X \to Y$ and $g : Y \to \mathbb{R}$, then $(g \circ f)^{-1}((-\infty, a]) = f^{-1}(g^{-1}(-\infty, a])$ for all $a \in \mathbb{R}$. Hence we have

Proposition 1.10. *If X and Y are topological space, $f : X \to Y$ is continuous and $g : Y \to \mathbb{R}$ is lower semicontinuous, then $g \circ f : X \to \mathbb{R}$ is lower semicontinuous.*

The following result is a direct consequence of Proposition 1.10.

Corollary 1.11. *For a metric space (X, d), if $f : [0, \infty) \to [0, \infty)$ is lower semicontinuous, then $f \circ d : X \times X \to [0, \infty)$ is lower semicontinuous.*

A *w-distance* on a metric space (X, d) is a function $p : X \times X \to [0, \infty)$ satisfying the following conditions :

(W1) $p(x, z) \leq p(x, y) + p(y, z)$ for all x, y, $z \in X$,

(W2) for each $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous,

(W3) for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all x, y, $z \in X$, $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The notion of w-distance on a metric space was first introduced by Shioji, Suzuki and Takahashi in [4]. In [5], various properties and examples of w-distances are given. In addition, some fixed point theorems are given in terms of w-distances.

We define Cauchy w-distances on a metric space naturally as follows : A w-distance p on a metric space (X, d) is called a *Cauchy w-distance* if every Cauchy sequence (x_n) in (X, d) has the property that for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $p(x_n, x_m) < \varepsilon$ for all $m > n \geq N$.

Example 1.12. (1) Let (X, d) be a metric space and $c > 0$. Define

$$p_1(x, y) = cd(x, y) \text{ and } p_2(x, y) = c \text{ for all } x,\ y \in X$$

Then p_1 and p_2 satisfy (W1)-(W3). It is clearly seen that p_1 is also Cauchy but p_2 is not (by using $\varepsilon = \frac{c}{2}$).

(2) Consider $\mathbb{R}$ as the metric space with usual metric. Define

$$p_3(x, y) = |y| \quad \text{for all } (x, y) \in X \times Y.$$

Then p_3 satisfies (W1)-(W3). Then p_3 is a w-distance on $\mathbb{R}$. Since $(x_n) = (1, 1, 1, \ldots)$ is a convergent sequence in $\mathbb{R}$ and $p_3(x_n, x_m) = 1$ for all $m, n \in \mathbb{N}$, it follows that p_3 is not Cauchy.

Our purpose is to consider when the metric $f \circ d : X \times X \to [0,\infty)$ is a w-distance and a Cauchy w-distance on (X,d) where (X,d) is a metric space and $f : [0,\infty) \to [0,\infty)$ is metric-preserving. It is shown that $f \circ d$ is a w-distance and a Cauchy w-distance on (X,d) if f is lower semicontinuous and continuous on $[0,\infty)$, respectively.

2 Main Results

The first main result is the following.

Theorem 2.1. *Let (X,d) be a metric space and $f : [0,\infty) \to [0,\infty)$ a metric-preserving function. If f is lower semicontinuous, then $f \circ d$ is a w-distance on (X,d).*

Proof. Since $f \circ d$ is a metric on X, we have that $f \circ d$ satisfies (W1). From Corollary 1.11, $f \circ d : X \times X \to [0,\infty)$ is lower semicontinuous. Then we have that $f \circ d(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous for all $x \in X$, that is, $f \circ d$ satisfies (W2).

To show that $f \circ d$ satisfies (W3), let $\varepsilon > 0$ be given. Since f is metric-preserving, by Proposition 1.9, there is a $\rho > 0$ such that

$$\text{for all } x \in [0,\infty),\ f(x) < \rho \Rightarrow x < \frac{\varepsilon}{2}. \tag{2.1}$$

Let $\delta = \dfrac{\rho}{2}$. Let x, y, $z \in X$ be such that $(f \circ d)(z,x) \le \delta$ and $(f \circ d)(z,y) \le \delta$. That is,

$$f(d(z,x)) < \rho \quad \text{and} \quad f(d(z,y)) < \rho. \tag{2.2}$$

We deduce from (1) and (2) that $d(z,x) < \dfrac{\varepsilon}{2}$ and $d(z,y) < \dfrac{\varepsilon}{2}$. Hence $d(x,y) \le d(x,z) + d(z,y) < \varepsilon$.

This proves that $f \circ d$ is a w-distance, as desired. □

Example 2.2. Let h be the metric-preserving function defined in Example 1.7(3). Clearly, h is lower semicontinuous but not continuous. If (X,d) is a metric space, then by Theorem 2.1,

$$(h \circ d)(x,y) = h(d(x,y)) = \begin{cases} 0 & \text{if } x = y, \\ c & \text{if } x \ne y, \end{cases}$$

is a w-distance on (X,d). If (X,d) has a Cauchy sequence (x_n) with $x_i \ne x_j$ if $i \ne j$, then $(h \circ d)(x_n,x_m) = c$ if $n \ne m$, and hence $h \circ d$ is not Cauchy.

Example 2.3. Let f be the metric-preserving function given in Example 1.8, that is,

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 + \frac{1}{x+1} & \text{if } x > 0. \end{cases}$$

Since f is strictly decreasing and continuous on $(0,\infty)$ and $\mathrm{Im} f = \{0\} \cup (1,2)$, it follows that

$$f^{-1}([0,a]) = \begin{cases} \{0\} & \text{if } \ a \le 1, \\ \{0\} \cup [f^{-1}(a),\infty) & \text{if } \ 1 < a < 2, \\ [0,\infty) & \text{if } \ a \ge 2. \end{cases}$$

Then f is lower semicontinuous. If (X,d) is a metric space, then by Theorem 2.1,

$$(f \circ d)(x,y) = \begin{cases} 0 & \text{if } \ x = y, \\ 1 + \frac{1}{d(x,y)+1} & \text{if } \ x \ne y, \end{cases}$$

is a w-distance on (X,d). Since $1+\frac{1}{d(x,y)+1} > 1$ for all distinct $x, y \in X$, we deduce that $f \circ d$ is not Cauchy if X contains a Cauchy sequence (x_n) with $x_i \ne x_j$ if $i \ne j$.

The next Theorem is our second main result.

Theorem 2.4. *Let (X,d) be a metric space and $f : [0,\infty) \to [0,\infty)$ a metric-preserving function. If f is continuous, then $f \circ d$ is a Cauchy w-distance on (X,d).*

Proof. It follows from Theorem 2.1 that $f \circ d$ is a w-distance on (X,d).

Next, let (x_n) be a Cauchy sequence in (X,d) and let $\varepsilon > 0$ be given. By Proposition 1.1, $f(0) = 0$. Since f is continuous at 0, there is a $\delta > 0$ such that

$$\text{for all } x \in [0,\infty),\ 0 \le x < \delta \ \Rightarrow\ f(x) < \varepsilon. \tag{2.1}$$

Since (x_n) is a Cauchy sequence in (X,d), there is an $N \in \mathbb{N}$ such that

$$\text{for all } \ m,n \in \mathbb{N},\ m,n \ge N \ \Rightarrow\ d(x_n,x_m) < \delta. \tag{2.2}$$

Hence (1) and (2) yield the result that

$$\text{for all } \ m,n \in \mathbb{N},\ m,n \ge N \ \Rightarrow\ (f \circ d)(x_n,x_m) \ = \ f(d(x_n,x_m)) \ < \ \varepsilon.$$

Therefore $f \circ d$ is a Cauchy w-distance on (X,d). □

Remark 2.5. In the proof of Theorem 2.4, the continuity of f at only 0 required. Proposition 1.3 tells us that the continuity of f at 0 and the continuity of f on $[0,\infty)$ are equivalent.

Example 2.6. Let $f, g : [0,\infty) \to [0,\infty)$ be the metric-preserving functions defined in Example 1.7(1) and (2), respectively, that is,

$$f(x) = \log_a(1+x) \ \text{ for all } x \in [0,\infty) \ \text{ where } a > 1,$$
$$g(x) = x^r \ \text{ for all } x \in [0,\infty) \ \text{ where } r > 0.$$

Then f and g are continuous on $[0, \infty)$. If (X, d) is a metric space, then by Theorem 2.4, both

$$(f \circ d)(x, y) = \log_a(1 + d(x, y)) \quad \text{for all } x, y \in X$$

and

$$(g \circ d)(x, y) = d(x, y)^r \quad \text{for all } x, y \in X$$

are Cauchy w-distances on (X, d).

In particular, $p, p' : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$p(x, y) = \log_a(1 + |x - y|) \text{ and } p'(x, y) = |x - y|^r \quad \text{for all } x,\ y \in \mathbb{R}$$

are Cauchy w-distances on $\mathbb{R}$.

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