ISSN 1686-0209

Thai Journal of Mathematics
Vol. 18, No. 1 (2020),
Pages 453 - 463

EXISTENCE AND ULAM STABILITY OF SOLUTION TO FRACTIONAL ORDER HYBRID DIFFERENTIAL EQUATIONS OF VARIABLE ORDER

Norravich Limpanukorn[1], **Parinya Sa Ngiamsunthorn**[2,*]
[1] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Thung Khru, Bangkok, 10140, Thailand*

Abstract In this paper, we study fractional hybrid differential equation of variable order $0 < \alpha(t) \leq 1$ in the sense of Caputo. The existence of solution is proved by Krasnoselkii fixed point theorem. In addition, we study stability of solutions in the sense of Ulam-Hyers stability. Finally, an example is given to illustrate the result.

MSC: 26A33; 34A08; 34A38

Keywords: Hybrid differential equations; Caputo fractional derivative; Krasnoselkii fixed point; Ulam-Hyers stability

Submission date: 01.12.2019 / Acceptance date: 08.02.2020

1. Introduction

Nonlinear systems caught much interest to engineers, biologists, physicists, mathematicians and many other scientists by various phenomena concealed in the nature. Nonlinear differential equations are used extensively in mathematical modelling to explain various systems. In recent decades, the development of differentiation and integration has been extended from integer order to arbitrary order which is known as fractional calculus. The area of fractional calculus becomes a vibrant research area due to the fact that some phenomena can be described more accurately with fractional derivative. There are several applications of fractional differential equation in areas of application, for example, economics, signal identification, image processing and so on (see [3, 9, 11, 12]). Several definitions of fractional derivatives are found in the literature such as Riemann-Liouville, Hadamard, Riesz, etc (see [1, 2]). One of the definitions was presented by Caputo and was studied by many experts (see [2, 13, 14, 16]). The Caputo fractional derivatives are widely used in physical application because the corresponding initial conditions involve integer order derivatives which reflect a conventional meaning.

*Corresponding author.

Published by The Mathematical Association of Thailand.

For the development in theory of fractional differential equations, the topics that caught much attention include the study on existence and uniquess of solutions, stability of solutions, multiplicity of solutions and positive solutions for boundary value problems. There are several techniques used to display the existence and uniqueness of solutions such as Banach fixed point theorem, upper-lower solution method, Schaefer fixed point theorem, Schauder fixed point theorem and so on (see [5, 7, 8, 10, 14, 17]). Furthermore, the perturbation of nonlinear differential equations appeals to several interests. In particular, the perturbation of the derivative term in a differential equation by subtracting a nonlinear term is known as a hybrid differential equation. The existence of solutions of the first-order hybrid differential equation given by

$$\begin{aligned} &\frac{d}{dt}[x(t) - f(t, x(t))] = g(t, x(t)), \quad \text{for} \quad t \in [t_0, t_0 + a] \\ &x(0) = x_0 \end{aligned}$$

was first studied (see [8]). In 2016, the problem was extended to fractional order derivative by [14] to study the existence and approximation of solutions to fractional order hybrid differential equation in the sense of Caputo derivative given by

$$\begin{aligned} &\frac{d^\alpha}{dt^\alpha}[x(t) - f(t, x(t))] = g(t, x(t)), \quad \text{for} \quad t \in [t_0, t_0 + a] \\ &x(0) = x_0. \end{aligned}$$

In 2018, [15] studied fractional diffrential equations involving a variable order RiemannLiouville fractional derivative of order $\alpha(t)$ given by

$$D_{a+}^{\alpha(t)} = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} x(\tau)\mathrm{d}\tau.$$

In 2019, [16] studied existence and unique result of approximate solutions to initial value problem for fractional differential equation of variable order of the form

$$\begin{aligned} &D_{0+}^{p(t)} x(t) = f(t, x, D_{0+}^{q(t)} x) \quad , 0 < t < \infty \\ &x(0) = 0, \end{aligned}$$

where the variable order Caputo fractional derivative was considered. Apart from the existence and uniqueness of solution, the Ulam-Stability of fractional order differential equation also caught much attention. In 2019, Ulam-stability for fractional initial value problem

$$\begin{aligned} &D_{0-}^{\alpha} u(t) = f(t, u(t)) \quad , t > 0 \\ &D_{0+}^{\alpha-1} u(0^+) = u_0 \end{aligned}$$

with Riemann fractional derivative was studied in [4].

Motivated by these works, the purpose of this research is to extend the study of existence and Ulam-stability of solution of Caputo fractional order hybrid differential equations to the one with variable order given by

$$\begin{aligned} &{}_0D_t^{\alpha(t)}[x(t) - f(t, x(t))] = g(t, x(t)), \quad \text{for} \quad t \in [0, T], 0 < \alpha(t) \le 1 \\ &x(0) = x_0 \end{aligned} \tag{1.1}$$

where

$${}_0D_t^{\alpha(t)} = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t (t-\tau)^{-\alpha(t)} x'(\tau)\mathrm{d}\tau.$$

Our result displays existence and stability of solutions to Caputo fractional order hybrid differential equation of one variable order which is more general than a differential equation with a noninteger constant order appeared in the literature.

This paper is designed as follows. In Section 2, the notation and concepts of fractional order hybrid differential equations of one variable order will be introduced and our framework will be discussed. The existence of solutions to equation (1.1) and their stability will be proved in Section 3 and Section 4, respectively. Lastly, in Section 5, an example will be given to illustrate the obtained results.

2. Preliminaries and Framework

Let $J = [0, T]$ be an interval in $\mathbb{R}$ where $T > 0$. We denote by $C(J, \mathbb{R})$ the space of continuous functions $x : J \to \mathbb{R}$. The space $C(J, \mathbb{R})$ is a Banach space with the supremum norm $\|.\|$ defined by

$$\|x\| = \sup_{t\in J} | x(t) |.$$

In this work, we consider the fractional order hybrid differential equation with initial condition given by (1.1), where $f \in C(J \times \mathbb{R}, \mathbb{R}), g \in C(J \times \mathbb{R}, \mathbb{R})$ and initial data $x_0 \in \mathbb{R}$. The fractional derivative involved in this study is in the sense of Caputo which is defined as follows.

Definition 2.1. [14] Let $\alpha > 0$, the left Caputo fractional derivative of order α for a function $x(t)$ is defined by

$${}_aD_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}x^{(n)}(\tau)\mathrm{d}\tau$$

where $n - 1 < \alpha < n$ and $n \in \mathbb{N}$.

Definition 2.2. [14] Let $\alpha > 0$, the left Riemann-Lioville fractional integral of order α for a function $x(t)$ is defined by

$${}_aI_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)\mathrm{d}\tau.$$

Definition 2.3. [1] Let $0 < \alpha(t) \leq 1$, the left Riemann-Lioville fractional integral of order $\alpha(t)$ for function $x(t)$ is defined by

$${}_aI_t^{\alpha(t)}x(t) = \frac{1}{\Gamma(\alpha(t))}\int_a^t (t-\tau)^{\alpha(t)-1}x(\tau)\mathrm{d}\tau, \quad t \in J.$$

Definition 2.4. [16] Let $0 < \alpha(t) < 1$, the left Caputo fractional derivative of order $\alpha(t)$ for function $x(t)$ is defined by

$${}_aD_t^{\alpha(t)}x(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_a^t (t-\tau)^{-\alpha(t)}x'(\tau)\mathrm{d}\tau, \quad t \in J.$$

Theorem 2.5. [1] Let $\alpha : [a, b] \to (n - 1, n]; n \in \mathbb{N}$, then

$${}_aI_t^{\alpha}\,{}_aD_t^{\alpha}x(t) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^k, \quad t \in [a, b].$$

Let $P = \{[0,T_1],(T_1,T_2],(T_2,T_3],...,(T_{N-1},T]\}$ where $P_k \in P$ is the k^{th} subinterval of P and let $\alpha : [0,T] \to (0,1]$ be a piecewise constant function with respect to P. Thus, we define the function $\alpha(t)$ as follow

$$\alpha(t) = \sum_{k=1}^{N} \alpha_k I_k(t) \quad ,t \in [0,T] \tag{2.1}$$

where $0 < \alpha_k \leq 1$, $k = 1,2,...,N$, and I_k is the indicator function on P_k, that is $I_k(t) = 1$ for $t \in P_k$, $I_k(t) = 0$ for elsewhere.
Thus the function $\alpha(t)$ can be written by

$$\alpha(t) = \begin{cases} \alpha_1 & ,t \in [0,T_1] \\ \alpha_2 & ,t \in (T_1,T_2] \\ \alpha_3 & ,t \in (T_2,T_3] \\ \vdots & \\ \alpha_N & ,t \in (T_{N-1},T]. \end{cases}$$

According to (2.1) we get

$$\int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} x'(s)ds = \sum_{k=1}^{N} I_k(t) \int_0^t \frac{(t-s)^{-\alpha_k}}{\Gamma(1-\alpha_k)} x'(s)ds. \tag{2.2}$$

so (1.1) can be written by

$$\sum_{k=1}^{N} I_k(t) \int_0^t \frac{(t-s)^{-\alpha_k}}{\Gamma(1-\alpha_k)} x'(s)ds - \sum_{k=1}^{N} I_k(t) \int_0^t \frac{(t-s)^{-\alpha_k}}{\Gamma(1-\alpha_k)} f'(s,x(s))ds = g(t,x(t)) \\ ,0 \leq t \leq T. \tag{2.3}$$

Hence, on the interval $[0,T_1]$, (2.3) satisfies

$$\int_0^t \frac{(t-s)^{-\alpha_1}}{\Gamma(1-\alpha_1)} x'(s)ds - \int_0^t \frac{(t-s)^{-\alpha_1}}{\Gamma(1-\alpha_1)} f'(s,x(s))ds = g(t,x(t)) \quad ,0 \leq t \leq T_1. \tag{2.4}$$

Again on the interval $(T_1,T_2]$, (2.3) satisfies

$$\int_0^t \frac{(t-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} x'(s)ds - \int_0^t \frac{(t-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} f'(s,x(s))ds = g(t,x(t)) \quad ,T_1 < t \leq T_2. \tag{2.5}$$

In the same way on the interval $(T_{i-1},T_i]$, $i = 3,4,5,...,N(T_N = T)$, (2.3) satisfies

$$\int_0^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} x'(s)ds - \int_0^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} f'(s,x(s))ds = g(t,x(t)) \quad ,T_{i-1} < t \leq T_i. \tag{2.6}$$

Now, we present the definition of solution to problem (1.1) , which is fundamental in our work.

Definition 2.6. [15] We say that (1.1) has a solution, if there exist $x_1 \in C[0,T_1]$ satisfying (2.4) and $x_1(0) = x_0$; $x_2 \in C[0,T_2]$ satisfying (2.5) and $x_2(0) = x_0$; $x_i \in C[0,T_i]$ satisfying (2.6) and $x_i(0) = x_0$ $(i = 3,4,...,N)$.

Definition 2.7. The function $x \in C(J,\mathbb{R})$ is called a solution of (1.1) if it satisfied the integral equation

$$ {}_0I_t^{\alpha(t)}\, {}_0D_t^{\alpha(t)}[x(t) - f(t,x(t))] = {}_0I_t^{\alpha(t)} g(t,x(t)) $$

From Theorem 2.5 and (2.1) the integral representation of a solution of equation (1.1) satisfies

$$ x_1(t) = x_0 + f(t,x_1(t)) - f(0,x_0)) + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1} g(s,x_1(s))\mathrm{d}s \tag{2.7} $$

on $[0,T_1]$.
Again the solution of equation (1.1) satisfies

$$ x_2(t) = x_0 + f(t,x_2(t)) - f(0,x_0)) + \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1} g(s,x_2(s))\mathrm{d}s \tag{2.8} $$

on $(T_1,T_2]$.
Repeatly, on the interval $(T_{i-1},T_i]$, we have

$$ x_i(t) = x_0 + f(t,x_i(t)) - f(0,x_0)) + \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1} g(s,x_i(s))\mathrm{d}s. \tag{2.9} $$

Let X be a Banach space. A mapping $A : X \to X$ is called a nonlinear contraction if there exists a continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$ \|Ax - Ay\| \le \phi(\|x-y\|), $$

for all $x, y \in X$, which $\phi(r) < r$ for $r > 0$. In particular if $\phi(r) = cr$, $0 \le c < 1$, then A is called a contraction on X with contraction constant c.

Theorem 2.8. [6] Let K be a closed, convex, and nonempty subset of X. Let $A, B : K \to X$ be two operators satisfying:
(a) $Ax + By \in K$ for all $x, y \in K$
(b) A is a contraction on K
(c) B is completely continuous on K
Then the operator equation $Ax + Bx = x$ has a solution.

Definition 2.9. Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a function. Then $f(t,u)$ is globally Lipshitz continuous if there is a constant $C > 0$ such that

$$ \|f(t,u) - f(t,v)\| \le C\|u-v\| $$

for all $x, y \in \mathbb{R}$ and all $t \in J$.

Definition 2.10. [4] The equation (1.1) is Ulam-Hyers stable if for each $\varepsilon > 0$ and for each solution $x_i \in C([0,T_i],\mathbb{R})$, $i = 1,2,\dots,N$ of the inequality

$$ |{}_0D_t^{\alpha_i}[x_i(t) - f(t,x_i(t))] - g(t,x_i(t))| \le \varepsilon \tag{2.10} $$

there exist a real number $c_i > 0$ and a solution $y_i \in C([0,T_i],\mathbb{R})$ of equation (1.1) with

$$ |x_i(t) - y_i(t)| \le c_i\varepsilon. $$

Definition 2.11. [4] The equation (1.1) is generalized Ulam-Hyers stable if for each $\varepsilon > 0$ and for each solution $x_i \in C([0,T_i],\mathbb{R})$, $i = 1,2,\dots,N$ of the inequality

$$|{}_0D_t^{\alpha_i}[x_i(t) - f(t,x_i(t))] - g(t,x_i(t))| \le \varepsilon \tag{2.11}$$

there exist a function $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ and a solution $y_i \in C([0,T_i],\mathbb{R})$ of (1.1) with

$$|x_i(t) - y_i(t)| \le \psi_i(\varepsilon).$$

3. Existence of solution

In this section, solution of fractional order hybrid differential equation of one variable order is obtained via Krasnoselkii fixed point theorem in a suitable function space. The initial value problem (1.1) can be reformulated as an integral representation of a solution given by

$$x_i(t) = x_0 + f(t,x_i(t)) - f(0,x_0) + \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1} g(s,x_i(s))\mathrm{d}s,$$

for $t \in (T_{i-1},T_i]$. We denote by $L^p(J,\mathbb{R})$ the space of all Lebesque integrable functions on J with $\|.\|_{L_p}$-norm given by

$$\|x\|_{L_p} = \left[\int_0^T |x(t)|^p \mathrm{d}t\right]^{\frac{1}{p}}.$$

Definition 3.1. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ said to satisfy L^q-Caratheodory's conditions if
(i) $t \to \beta(t,x)$ is measurable for each $x \in \mathbb{R}$
(ii) $x \to \beta(t,x)$ is continuous a.e. $t \in J$ and
(iii) for each $r > 0$, there exist function $h_r \in L^q(J,\mathbb{R})$ such that

$$|\beta(t,x)| \le h_r(t), \quad t \in J, x \in \mathbb{R}, |x| \le r.$$

We state the assumptions used in this paper as follows:
(A0) There exist positive constants $M_f < 1$ and $\tilde{M}_f < 1$ such that

$$\|f(t,u) - f(t,v)\| \le M_f\|u - v\| \quad \text{and} \quad |f(t,u)| \le \tilde{M}_f(\|u\| + 1),$$

for all $u, v \in C(J,\mathbb{R})$.
(A1) The function $g(t,x)$ is L^q-Caratheodory.
(A2) There exist positive constants M_g such that

$$\|g(t,u) - g(t,v)\| \le M_g\|u - v\|,$$

for all $u, v \in C(J,\mathbb{R})$.

Theorem 3.2. Suppose assumptions (A0)-(A1) are satisfied, then the fractional hybrid differential equation with variable order (1.1) has a solution if $\alpha(t) \in (\frac{1}{q}, 1]$.

Proof. For each $i = 1,2,\dots,N$, we define $A, B : C([0,T_i],\mathbb{R}) \to C([0,T_i],\mathbb{R})$ by

$$Ax(t) = x_0 + f(t,x(t)) - f(0,x_0)$$

$$Bx(t) = \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1} g(s,x(s))\mathrm{d}s.$$

We set $B_R(x_0) := \{x \in C([0,T_i],\mathbb{R}) | |x - x_0| \leq R\}$. Clearly $B_R(x_0) \subseteq C([0,T_i],\mathbb{R})$ and $B_R(x_0)$ is closed bounded subset of Banach space.
We first show that $Ax + By \in B_R(x_0)$ when $x, y \in B_R(x_0)$. For any $t \in [0,T_i]$, we apply Hölder inequality to get

$$
\begin{aligned}
|Ax(t) + By(t) - x_0| &\leq |f(t,x(t)) - f(0,x_0)| \\
&\quad + \frac{1}{\Gamma(\alpha_i)} \left(\int_0^t |(t-s)^{\alpha_i - 1}|^p \mathrm{d}s\right)^{\frac{1}{p}} \left(\int_0^t |g(s,x(s))|^q \mathrm{d}s\right)^{\frac{1}{q}} \\
&\leq \tilde{M}_f(\|x\| + 1) + \tilde{M}_f(|x_0| + 1) \\
&\quad + \frac{1}{\Gamma(\alpha_i)} \left(\int_0^t |(t-s)^{\alpha_i - 1}|^p \mathrm{d}s\right)^{\frac{1}{p}} \left(\int_0^t |g(s,x(s))|^q \mathrm{d}s\right)^{\frac{1}{q}} \\
&\leq \tilde{M}_f(|x_0| + R + 1) + \tilde{M}_f(|x_0| + 1) \\
&\quad + \frac{1}{\Gamma(\alpha_i)} \left(\int_0^t |(t-s)^{\alpha_i - 1}|^p \mathrm{d}s\right)^{\frac{1}{p}} \left(\int_0^t |g(s,x(s))|^q \mathrm{d}s\right)^{\frac{1}{q}} \\
&\leq R\tilde{M}_f + 2\tilde{M}_f(\|x_0\| + 1) + \frac{1}{\Gamma(\alpha_i)} \left(\frac{t^{\alpha_i p - p + 1}}{\alpha_i p - p + 1}\right)^{\frac{1}{p}} \|h_r\|_q \\
&\leq R\tilde{M}_f + 2\tilde{M}_f(\|x_0\| + 1) + \frac{1}{\Gamma(\alpha_i)} \left(\frac{T^{\alpha_i p - p + 1}}{\alpha_i p - p + 1}\right)^{\frac{1}{p}} \|h_r\|_q \\
&\leq R\tilde{M}_f + 2\tilde{M}_f(\|x_0\| + 1) + \frac{1}{\Gamma(\alpha_i)} \left(\frac{T^{\frac{q}{q-1}\alpha_i - \frac{1}{q-1}}}{\frac{q}{q-1}\alpha_i - \frac{1}{q-1}}\right)^{\frac{q-1}{q}} \|h_r\|_q \\
&< R
\end{aligned}
$$

when R is large enough and $\frac{1}{p} + \frac{1}{q} = 1$. Hence, $Ax + By \in B_R(x_0)$.

Next, we show that A is contraction on $B_R(x_0)$. Clearly from Assumption (A0), we have

$$
\begin{aligned}
\|Ax(t) - Ay(t)\| &= \|f(t,x(t)) - f(t,y(t))\| \\
&\leq M_f \|x - y\|.
\end{aligned}
$$

It follows that A is a contraction on $B_R(x_0)$.

Let $\{x_n\}$ be a bounded sequence of functions in $C([0,T_i],\mathbb{R})$ such that $\|x_n\| \leq r$ for all $n = 1,2,3,\ldots$. By Hölder inequality and (A1) we get

$$
\begin{aligned}
\|Bx_n\| &\leq \sup_{t\in[0,T_i]} \frac{1}{\Gamma(\alpha_i)} \int_0^t \left|(t-s)^{\alpha_i - 1} g(s,x_n(s))\right| \mathrm{d}s \\
&\leq \sup_{t\in[0,T_i]} \frac{1}{\Gamma(\alpha_i)} \left(\int_0^t |(t-s)^{\alpha_i - 1}|^p \mathrm{d}s\right)^{\frac{1}{p}} \left(\int_0^t |g(s,x_i(s))|^q \mathrm{d}s\right)^{\frac{1}{q}} \\
&\leq \sup_{t\in[0,T_i]} \frac{1}{\Gamma(\alpha_i)} \left(\frac{t^{\alpha_i p - p + 1}}{\alpha_i p - p + 1}\right)^{\frac{1}{p}} \|h_r\|_q \\
&\leq \frac{1}{\Gamma(\alpha_i)} \left(\frac{T^{\frac{q}{q-1}\alpha_i - \frac{1}{q-1}}}{\frac{q}{q-1}\alpha_i - \frac{1}{q-1}}\right)^{\frac{q-1}{q}} \|h_r\|_q
\end{aligned}
$$

Since $\alpha_i \in (\frac{1}{q}, 1]$, it follows that $\frac{q}{q-1}\alpha_i - \frac{1}{q-1} > 0$. This show that $B(B_R(x_0))$ is uniformly bounded in $C([0, T_i], \mathbb{R})$.

Next we will show that the sequence is equicontinuous. Let $t_1, t_2 \in [0, T_i]$ which $t_1 \leq t_2$

$$
\begin{aligned}
&| Bx_n(t_1) - Bx_n(t_2) | \\
&\leq \frac{1}{\Gamma(\alpha_i)} \left| \int_0^{t_1} (t_1 - s)^{\alpha_i - 1} g(s, x_n(s)) \mathrm{d}s - \int_0^{t_2} (t_2 - s)^{\alpha_i - 1} g(s, x_n(s)) \mathrm{d}s \right| \\
&\leq \frac{1}{\Gamma(\alpha_i)} \left| \int_0^{t_1} \left[(t_1 - s)^{\alpha_i - 1} - (t_2 - s)^{\alpha_i - 1} \right] g(s, x_n(s)) \mathrm{d}s \right| \\
&\quad + \frac{1}{\Gamma(\alpha_i)} \left| \int_{t_1}^{t_2} (t_2 - s)^{\alpha_i - 1} g(s, x_n(s)) \mathrm{d}s \right| \\
&\leq \frac{1}{\Gamma(\alpha_i)} \int_0^{t_1} \left| (t_1 - s)^{\alpha_i - 1} - (t_2 - s)^{\alpha_i - 1} \right| h_r(s) \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha_i)} \int_{t_1}^{t_2} \left| (t_2 - s)^{\alpha_i - 1} g(s, x_n(s)) \right| \mathrm{d}s \\
&\to 0,
\end{aligned}
$$

as $|t_1 - t_2| \to 0$. Since $B(B_R(x_0))$ is equicontinuous, we apply Arzela-Ascoli theorem to claim that $B(B_R(x_0))$ is relatively compact in $B_R(x_0)$. Since this result holds for each interval $[0, T_i]$, $i = 1, 2, ..., N$, we obtain the existence of solution to (1.1) from the Krasnoselkii fixed point theorem (Theorem 2.8). ■

4. Ulam-Hyers stability

From the integral representation of solution of (1.1) for each $t \in P_i$ is given by (2.9), we have the following stability result.

Theorem 4.1. Suppose that assumptions (A0)-(A2) is satisfied. Then the fractional hybrid differential equation (1.1) is Ulam-Hyers stable.

Proof. Let $v_i \in C([0, T_i], \mathbb{R})$ satisfies the inequality (2.10), $i = 1, 2, \ldots, N$. It follows that

$$
\begin{aligned}
&\left| v_i(t) - v_i(0) - f(t, v_i(t)) + f(0, v_i(0)) - \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 1} g(s, v_i(s)) \mathrm{d}s \right| \\
&\quad \leq \varepsilon \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 1} \mathrm{d}s \\
&\quad \leq \varepsilon \frac{1}{\Gamma(\alpha_i + 1)} t^{\alpha_i} \\
&\quad \leq \varepsilon \frac{T^{\alpha_i}}{\Gamma(\alpha_i + 1)}
\end{aligned}
$$

We let $u_i \in C([0, T_i], \mathbb{R})$, $i = 1, 2, \ldots, N$ be a solution of

$$
\begin{aligned}
&{}_0D_t^{\alpha_i}[x(t) - f(t, x(t))] = g(t, x(t)), \quad \text{for} \quad t \in [0, T_i], \\
&x(0) = v_i(0).
\end{aligned} \tag{4.1}
$$

Hence, it satisfies

$$
u_i(t) = v_i(0) + f(t, u_i(t)) - f(0, v_i(0))) + \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 1} g(s, u_i(s)) \mathrm{d}s.
$$

So we obtain from (A0) and (A2)

$$
\begin{aligned}
&|v_i(t) - u_i(t)| \\
&= \left| v_i(t) - v_i(0) - f(t, u_i(t)) + f(0, v_i(0))) - \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} g(s, u_i(s)) \mathrm{d}s \right| \\
&\leq \left| v_i(t) - v_i(0) - f(t, v_i(t)) + f(0, v_i(0))) - \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} g(s, v_i(s)) \mathrm{d}s \right| \\
&\quad + |f(t, v_i(t)) - f(t, u_i(t))| + \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} |g(s, v_i(s)) - g(s, u_i(s))| \, \mathrm{d}s \\
&\leq \varepsilon \frac{T^{\alpha_i}}{\Gamma(\alpha_i + 1)} + M_f |v_i(t) - u_i(t)| + \frac{M_g}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} |v_i(s) - u_i(s)| \, \mathrm{d}s.
\end{aligned}
$$

Since $M_f < 1$, we can write

$$
\begin{aligned}
&|v_i(t) - u_i(t)| \\
&\leq \varepsilon \frac{T^{\alpha_i}}{(1 - M_f)\Gamma(\alpha_i + 1)} + \frac{M_g}{(1 - M_f)\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} |v_i(s) - u_i(s)| \, \mathrm{d}s.
\end{aligned}
$$

By Gronwall inequality, we obtain

$$
\begin{aligned}
&|v_i(t) - u_i(t)| \\
&\leq \varepsilon \frac{T^{\alpha_i}}{(1 - M_f)\Gamma(\alpha_i + 1)} \exp\left\{ \frac{M_g}{(1 - M_f)\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} \mathrm{d}s \right\} \\
&\leq \varepsilon \frac{T^{\alpha_i}}{(1 - M_f)\Gamma(\alpha_i + 1)} \exp\left\{ \frac{M_g T^{\alpha_i}}{(1 - M_f)\Gamma(\alpha_i + 1)} \right\}.
\end{aligned}
$$

Therefore, (1.1) is Ulam-Hyers stable. ■

5. Example

In this section, we give an example of hybrid differential equation of variable order to illustrate our result. Consider the following hybrid fractional differential equation of variable order

$$
\begin{aligned}
{}_0D_t^{\alpha(t)}[x(t) - f(t, x(t))] &= \frac{\tan^{-1} x(t)}{12}, \quad \text{for } \; t \in [0, 6] \\
x(0) &= 0
\end{aligned}
\tag{5.1}
$$

where

$$
f(t, x(t)) = \frac{x(t)}{2(x^2(t) + 1)} \tag{5.2}
$$

and the fractional order $\alpha(t)$ defined by

$$
\alpha(t) = \begin{cases} \dfrac{3}{4} & , t \in [0, 2] \\ \dfrac{3}{5} & , t \in (2, 4] \\ \dfrac{4}{5} & , t \in (4, 6]. \end{cases}
$$

It can be seen that $g(t,x) = \frac{\tan^{-1} x}{12}$ satisfies L^2-Caratheodory condition since $|g(t,x)| \leq \frac{\pi}{24}$, and the order $\alpha_i \in (\frac{1}{2}, 1]$. So the assumption (A1) is satisfied. Moreover, by mean value theorem, we see that

$$|g(t,x) - g(t,y)| = \left| \frac{\tan^{-1}(x)}{12} - \frac{\tan^{-1}(y)}{12} \right| \leq \frac{1}{12}|x - y|,$$

for $x, y \in \mathbb{R}$. Hence, (A2) is satisfied.

For the assumption on f, we can see that

$$\frac{d}{dx}\left(\frac{x}{2(x^2+1)}\right) = \frac{1-x^2}{2(x^2+1)^2}$$

and

$$\left|\frac{1-x^2}{2(x^2+1)^2}\right| \leq \frac{1}{2}$$

for all $x \in \mathbb{R}$. By mean value theorem we get

$$\begin{aligned} |f(t,x) - f(t,y)| &= \left| \frac{x}{2(x^2+1)} - \frac{y}{2(y^2+1)} \right| \\ &\leq \frac{1}{2}\|x - y\| \end{aligned}$$

Since $f(t,0) = 0$, it follows that

$$|f(t,x)| = |f(t,x) - f(t,0)| \leq \frac{1}{2}|x| \leq \frac{1}{2}(|x|+1),$$

which means that assumption (A0) holds with $M_f = \tilde{M}_f = \frac{1}{2}$. Therefore, all asuumptions in Theorem 3.2 and Theorem 4.1 are satisfied. Hence, the hybrid fractional differential equation of variable order (5.1) has a solution and is Ulam-Hyers stable. We remark that the solution of (5.1) satisfies

$$\begin{aligned} x_1(t) &= \frac{x_1(t)}{2(x_1^2(t)+1)} + \frac{1}{12\Gamma(\frac{3}{4})}\int_0^t (t-s)^{-\frac{1}{4}}\tan^{-1}(x_1(s))\mathrm{d}s && \text{on } t \in [0,2] \\ x_2(t) &= \frac{x_2(t)}{2(x_2^2(t)+1)} + \frac{1}{12\Gamma(\frac{3}{5})}\int_0^t (t-s)^{-\frac{2}{5}}\tan^{-1}(x_2(s))\mathrm{d}s && \text{on } t \in (2,4] \\ x_3(t) &= \frac{x_3(t)}{2(x_3^2(t)+1)} + \frac{1}{12\Gamma(\frac{4}{5})}\int_0^t (t-s)^{-\frac{1}{5}}\tan^{-1}(x_3(s))\mathrm{d}s && \text{on } t \in (4,6]. \end{aligned}$$

6. Conclusion

We present the proof of existence and stability of solutions for fractional order hybrid diffrential equations with varible order Caputo fractional derivative $\alpha(t)$. The existence result is proved by using Krasnoselskii fixed point theorem under Caratheodory and Lipschitz conditions for nonlinear terms. The result extends the class of fractional differential equations to the one with variable order which is illustrated by an example.

Acknowledgements

The authors would like to thanks referees for useful comments and suggestions which lead to an extensive improvement of the manuscripts.

References

[1] R. Almeida, D. Tavares, D.F.M. Torres, The Variable-Order Fractional Calculus of Variations; Springer International Publishing, New York, 2018.

[2] G.A. Anastassiou, On right fractional calculus. Chaos, Solutions and Fractals 42(1)(2009) 365–376.

[3] A. Atangana, Convergence and stability analysis of a novel iteration method for fractional biological population equation, Neural Comput & Applic. 25(5)(2014) 1021–1030.

[4] M. Benchohra, S. Bouriah, J. Nieto, Existence and Ulam stability for nonlinear implicit differential equations with Riemann-Liouville fractional derivative, Demonstratio Mathematica 52(1)(2019) 437–450.

[5] P. Borisut, K. Khammahawong, P. Kumam, Fixed Point Theory Approach to Existence of Solutions with Differential Equations; IntechOpen, London, 2018.

[6] T.A. Burton, A Fixed-point theorem of Krasnoselskii, Appl. Math. Lett. 11(1)(1998) 85–88.

[7] B.C. Dhage, On a Fixed Point Theorem of Krasnoselskii-Schaeer Type, E.J.Q.T.D.E. 6(2002) 1–9.

[8] B.C. Dhage, S.B. Dhage, S.K. Ntouyas, Approximating solutions of nonlinear hybrid differential equations, Appl. Math. Lett. 34(2014) 76–80.

[9] R. Hilfer, Applications of Fractional Calculus in Physics. World Scientic; Singapore, 2000.

[10] R.A. Khan, K. Shah, Existence and uniqueness of positive solutions to fractional order multi-point boundary value problems, Commun. Appl. Anal. 19(2015) 515–526.

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations. Elsevier Science Inc., New York, 2006.

[12] K.B. Oldham, J. Spanier, The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order; AP, New York, 1974.

[13] M.A. Özarslan, C. Ustaoglu, Incomplete Caputo fractional derivative operators. Adv. Differ. Equ. 2018 (2018) 209.

[14] D. Somjaiwang, P. Sa Ngiamsunthorn, Existence and approximation of solutions to fractional order hybrid differential equations, Adv. Differ. Equ. 2016 (2016) 278–288.

[15] S. Zhang, The uniqueness result of solutions to initial value problems of differential equations of variable-order, RACSAM. 112(2)(2018) 407–423.

[16] S. Zhang , L. Hu, Unique Existence Result of Approximate Solution to Initial Value Problem for Fractional Differential Equation of Variable Order Involving the Derivative Arguments on the Half-Axis, Mathematics 7(2019) 286–309.

[17] Y. Zhao, S. Sun, Z. Han, Q. Li, Theory of fractional hybrid dierential equations, Comput. Math. Appl. 62(2011) 1312–1324.