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EXPLICIT AND IMPLICIT ITERATIVE SCHEMES WITH BALANCED MAPPINGS IN HADAMARD SPACES

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Abstract We obtain two different kinds of approximation theorems to a common fixed point of a finite family of nonexpansive mappings defined on a Hadamard space. One is an explicit iterative method and the other is an implicit iterative method. We use the idea of balanced mappings to generate the approximate sequences.

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1. Introduction

Approximation of a common fixed point of a family of nonlinear mappings is one of the central topics in fixed point theory and a number of researchers have been dealing with this problem. They have been studying several different methods for generating an approximate sequence, and one of the most popular scheme is so called a Mann type [8] iterative method. Namely, for a given initial point x_1, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$$

for $n \in \mathbb{N}$. Reich [10] proved that this sequence converges weakly to a fixed point of a nonexpansive mapping T defined on a closed convex subset of Banach space under certain assumptions. Since then, this scheme has been studied intensively and a great variety of generalized results have been proved. In particular, we focus on its generalization to a complete $\mathrm{CAT}(\kappa)$ space given by He, Fang, Lopez, and Li [6].

Theorem 1.1 (He, Fang, Lopez, and Li [6]). *Let X be a complete $\mathrm{CAT}(\kappa)$ space. Let $T : X \to X$ be a nonexpansive mapping such that the set $\operatorname{Fix} T$ of fixed points of T is nonempty. Let $x_1 \in X$ be such that $d(x_1, \operatorname{Fix} T) < D_\kappa/4$, where $D_\kappa = \infty$ if $\kappa \le 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$. For a sequence of positive real numbers $\{\alpha_n\} \subset]0, 1[$ satisfying $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n)Tx_n$$

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for $n \in \mathbb{N}$*. Then,* $\{x_n\}$ *is* Δ*-convergent to some point in* $\operatorname{Fix} T$*.*

Recently, a Δ-convergence theorem for a finite family of nonexpansive mappings, which is analogous result to the theorem above, was proved by Hasegawa and Kimura [5]. They considered the mappings defined on a Hadamard space, a complete CAT(0) space, and used the notion of balanced mapping to generate an iterative sequence.

On the other hand, Xu and Ori [12] proposed another kind of iterative scheme converging weakly to a common fixed point of a finite family of nonexpansive mappings in the setting of a Hilbert space, which is called an implicit iterative method.

Theorem 1.2 (Xu and Ori [12])**.** *Let* C *be a closed convex subset of a Hilbert space and* $T_k : C \to C$ *a nonexpansive mapping for* $k = 1, 2, \dots, N$ *with* $\bigcap_{k=1}^{N} \operatorname{Fix} T_k \neq \emptyset$*. For a positive real sequence* $\{\alpha_n\} \subset]0, 1[$ *converging to* 0 *and for given* $x_1 \in C$*, generate* $\{x_n\}$ *by the following implicit iterative formula:*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{(n \bmod N)+1} x_{n+1}$$

for $n \in \mathbb{N}$*. Then* $\{x_n\}$ *is well defined and convergent weakly to some point in* $\bigcap_{k=1}^{N} \operatorname{Fix} T_k$*.*

In this paper, we consider these two kinds of iterative methods for a finite family of nonexpansive mappings defined on a Hadamard space. We use the notion of balanced mapping to generate iterative sequences, and obtain their Δ-convergence to a common fixed point of mappings. We note that the iterative method we propose is different from that in [5]; we do not use any convex combination to generate sequences, and instead of it, we only use the notion of balanced mappings.

2. Preliminaries

Let X be a metric space with a metric d. For $x, y \in X$ and $l \geq 0$, a mapping $c : [0, l] \to X$ is called a geodesic with endpoints $x, y \in X$ if it satisfies that $c(0) = x$, $c(l) = y$, and $d(c(t), c(s)) = |t - s|$ for every $t, s \in [0, l]$. If a geodesic with endpoints x and y exists for all $x, y \in X$, we say X to be a geodesic metric space. In what follows, we assume that X has a unique geodesic for every $x, y \in X$. Then, we denote the image of the geodesic with endpoints $x, y \in X$ by $[x, y]$, which is well defined.

A geodesic triangle $\triangle(x, y, z)$ with vertices $x, y, z \in X$ is defined as the union of three segments $[x, y]$, $[y, z]$, and $[z, x]$. Its comparison triangle $\triangle(\overline{x}, \overline{y}, \overline{z})$ is defined as the triangle in the 2-dimensional Euclidean space $\mathbb{R}^2$ whose length of each corresponding edge is identical with that of the original triangle;

$$d(y, z) = \|\overline{y} - \overline{z}\|, \; d(z, x) = \|\overline{z} - \overline{x}\|, \; d(x, y) = \|\overline{x} - \overline{y}\|,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = \|\overline{x} - \overline{p}\|$. If for any $p, q \in \triangle(x, y, z)$ and their comparison points $\overline{p}, \overline{q} \in \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$, the inequality

$$d(p, q) \leq \|\overline{p} - \overline{q}\|$$

holds for all triangles in X, then we call X a CAT(0) space. A Hadamard space is defined as a complete CAT(0) space.

For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1 - t) d(x, y)$ and $d(z, y) = t d(x, y)$. We denote it by $tx \oplus (1 - t) y$. A subset C of X is said to be convex if $tx \oplus (1-t)y \in C$ for every $x, y \in C$ and $t \in [0,1]$. In a Hadamard space X, the following important inequalities holds:

$$d(z, tx \oplus (1-t)y)^2 \leq td(z,x)^2 + (1-t)d(z,y)^2 - t(1-t)d(x,y)^2$$

for every $x, y, z \in X$ and $t \in [0,1]$;

$$d(x,u)^2 + d(y,v)^2 - d(x,y)^2 - d(u,v)^2 \leq 2d(x,v)d(y,u)$$

for every $u, v, x, y \in X$. For the basic properties of Hadamard spaces, see [1, 2].

A mapping $T : X \to X$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x,y)$ for every $x, y \in X$. We know that the set $\operatorname{Fix} T = \{z \in X : z = Tz\}$ of all fixed points of nonexpansive mapping T is closed and convex.

For a bounded sequence $\{x_n\}$ in X, let $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$ for $x \in X$, and define the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ by

$$r(\{x_n\}) = \inf_{x\in X} r(x, \{x_n\}).$$

The asymptotic center of $\{x_n\}$ is a set of point $p \in X$ satisfying that $r(p, \{x_n\}) = r(\{x_n\})$. Denoting the set of all minimizers of a function $f : X \to \mathbb{R} \cup \{\infty\}$ by $\operatorname{argmin}_{x\in X} f(x)$, we can say that the set of all asymptotic centers of $\{x_n\}$ is $\operatorname{argmin}_{x\in X} r(x, \{x_n\})$. We know that an asymptotic center of $\{x_n\}$ consists of exactly one point [3]. We say that $\{x_n\}$ is Δ-convergent to $x_0 \in X$ if x_0 is the unique asymptotic center of any subsequence of $\{x_n\}$. We know that every bounded sequence in a Hadamard space has a Δ-convergent subsequence; see [4, 7].

The following theorem [5] shows the definition and fundamental properties of the generalized convex combination of a finite family of nonexpansive mappings, which is also called a balanced mapping.

Theorem 2.1 (Hasegawa and Kimura [5]). *Let X be a Hadamard space and $T_k : X \to X$ a nonexpansive mapping for $k = 0, 1, 2, \dots, N$. Suppose that a finite family $\{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_N\}$ of positive real numbers satisfies $\sum_{k=0}^N \alpha_k = 1$. For each $x \in X$, define a subset*

$$U(x) = \operatorname*{argmin}_{y\in X} \sum_{k=0}^{N} \alpha_k d(T_k x, y)^2$$

of X. Then the following holds:

(i) *$U(x)$ consists of one point for every $x \in X$ and thus $U : X \to X$ can be defined as a single-valued mapping.*
(ii) *U is nonexpansive;*
(iii) *if $\bigcap_{k=1}^N \operatorname{Fix} T_k$ is nonempty, then it coincides with $\operatorname{Fix} U$.*

3. Explicit and implicit iterative schemes

In this section, we prove two Δ-convergence theorems to a common fixed point of a finite family of nonexpansive mappings. The first one is an iterative scheme defined by an explicit form of recurrence formula.

Theorem 3.1. *Let X be a Hadamard space. Let $T_k : X \to X$ be a nonexpansive mapping for $k = 1, 2, \dots, N$ and suppose that $\bigcap_{k=1}^N \operatorname{Fix} T_k \neq \emptyset$. Let $\{\alpha_n^k\}$ be real sequences for $k = 0, 1, 2, \dots, N$ and $a, b \in \mathbb{R}$ satisfying the following conditions:*

(i) *$0 < a \leq \alpha_n^k \leq b < 1$ for all $k = 0, 1, 2, \dots, N$ and $n \in \mathbb{N}$;*

(ii) $\sum_{k=0}^{N} \alpha_n^k = 1$ *for all* $n \in \mathbb{N}$.

For given $x_1 \in X$, *generate a sequence* $\{x_n\} \subset X$ *by*

$$x_{n+1} = \operatorname*{argmin}_{y \in X} \left(\alpha_n^0 d(x_n, y)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, y)^2 \right)$$

for $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *is* Δ*-convergent to some* $x_0 \in \bigcap_{k=1}^{N} \operatorname{Fix} T_k$.

Proof. For each $n \in \mathbb{N}$, define a mapping $U_n : X \to X$ by

$$U_n x = \operatorname*{argmin}_{y \in X} \left(\alpha_n^0 d(x, y)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x, y)^2 \right)$$

for $x \in X$. Since the identity mapping is also nonexpansive and the set of its fixed points is the whole space, by Theorem 2.1, we get U_n is nonexpansive and $\operatorname{Fix} U_n = \bigcap_{k=1}^{N} \operatorname{Fix} T_k$. We also have $x_{n+1} = U_n x_n$ for $n \in \mathbb{N}$. Then, for $p \in \bigcap_{k=1}^{N} \operatorname{Fix} T_k$, we have

$$d(x_{n+1}, p) = d(U_n x_n, p) \leq d(x_n, p)$$

for all $n \in \mathbb{N}$. It follows that a nonnegative real sequence $\{d(x_n, p)\}$ is nonincreasing and thus it is convergent to some $c_p \in \mathbb{R}$ as $n \to \infty$. We also have $\{x_n\}$ is bounded. On the other hand, for $p \in \bigcap_{k=1}^{N} \operatorname{Fix} T_k$, we have

$$\begin{aligned}
&\alpha_n^0 d(x_n, U_n x_n)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, U_n x_n)^2 \\
&\leq \alpha_n^0 d(x_n, tU_n x_n \oplus (1-t)p)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, tU_n x_n \oplus (1-t)p)^2 \\
&\leq \alpha_n^0 \left(td(x_n, U_n x_n)^2 + (1-t)d(x_n, p)^2 - t(1-t)d(U_n x_n, p)^2 \right) \\
&\quad + \sum_{k=1}^{N} \alpha_n^k \left(td(T_k x_n, U_n x_n)^2 + (1-t)d(T_k x_n, p)^2 - t(1-t)d(U_n x_n, p)^2 \right) \\
&\leq \alpha_n^0 \left(td(x_n, U_n x_n)^2 + (1-t)d(x_n, p)^2 - t(1-t)d(x_{n+1}, p)^2 \right) \\
&\quad + \sum_{k=1}^{N} \alpha_n^k \left(td(T_k x_n, U_n x_n)^2 + (1-t)d(x_n, p)^2 - t(1-t)d(x_{n+1}, p)^2 \right) \\
&\leq t \left(\alpha_n^0 d(x_n, U_n x_n)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, U_n x_n)^2 \right) \\
&\quad + (1-t)d(x_n, p)^2 - t(1-t)d(x_{n+1}, p)^2,
\end{aligned}$$

and thus

$$\begin{aligned}
(1-t) &\left(\alpha_n^0 d(x_n, U_n x_n)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, U_n x_n)^2 \right) \\
&\leq (1-t)d(x_n, p)^2 - t(1-t)d(x_{n+1}, p)^2.
\end{aligned}$$

Dividing both sides by $1-t$ and tending $t \to 1$, we get

$$\alpha_n^0 d(x_n, U_n x_n)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, U_n x_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

It follows that

$$ad(x_n, U_n x_n)^2 \le \alpha_n^0 d(x_n, U_n x_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2 \to c_p^2 - c_p^2 = 0.$$

Thus we have $d(x_n, U_n x_n) \to 0$ as $n \to \infty$. In the same fashion, we get

$$\begin{aligned}(1-b)ad(T_k x_n, U_n x_n)^2 &\le \alpha_n^k d(T_k x_n, U_n x_n)^2 \\ &\le d(x_n, p)^2 - d(x_{n+1}, p)^2 \\ &\to c_p^2 - c_p^2 = 0\end{aligned}$$

for $k = 1, 2, \dots, N$ and we have $d(T_k x_n, U_n x_n) \to 0$ as $n \to \infty$. From these facts, we obtain

$$d(x_n, T_k x_n) \le d(x_n, U_n x_n) + d(U_n x_n, T_k x_n) \to 0$$

as $n \to \infty$ for every $k = 1, 2, \dots, N$. Let $x_0 \in X$ be a unique asymptotic center of a bounded sequence $\{x_n\}$ and we will show that an asymptotic center $u \in X$ of any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ is identical to x_0. We know that u belongs to $\bigcap_{k=1}^N \operatorname{Fix} T_k$. Indeed, since $\{d(x_{n_i}, T_k x_{n_i})\}$ converges to 0, from the definition of asymptotic center we have

$$\begin{aligned}r(\{x_{n_i}\}) &= \limsup_{i\to\infty} d(x_{n_i}, u) \\ &\le \limsup_{i\to\infty} d(x_{n_i}, T_k u) \\ &\le \limsup_{i\to\infty} (d(x_{n_i}, T_k x_{n_i}) + d(T_k x_{n_i}, T_k u)) \\ &= \limsup_{i\to\infty} d(T_k x_{n_i}, T_k u) \\ &\le \limsup_{i\to\infty} d(x_{n_i}, u) = r(\{x_{n_i}\})\end{aligned}$$

for all $k = 1, 2, \dots, N$. This implies that $T_k u$ is an asymptotic center of $\{x_{n_i}\}$, and from the uniqueness of an asymptotic center, we get $u = T_k u$ for $k = 1, 2, \dots, N$, that is, $u \in \bigcap_{k=1}^N \operatorname{Fix} T_k$. It follows that $\{d(x_n, u)\}$ is convergent to c_u. Therefore, we have

$$\begin{aligned}r(\{x_n\}) &= \limsup_{n\to\infty} d(x_n, x_0) \\ &\le \limsup_{n\to\infty} d(x_n, u) = c_u = \lim_{i\to\infty} d(x_{n_i}, u) \\ &\le \limsup_{i\to\infty} d(x_{n_i}, x_0) \\ &\le \limsup_{n\to\infty} d(x_n, x_0) = r(\{x_n\}).\end{aligned}$$

Thus u is identical to x_0. Hence, by definition, $\{x_n\}$ is Δ-convergent to $x_0 \in \bigcap_{k=1}^N \operatorname{Fix} T_k$, which is the desired result. ■

We remark that the iterative scheme used in this theorem corresponds to the following sequence in the case where the underlying space X is a closed convex subset of a Hilbert space: $x_1 \in X$ and $x_{n+1} = \alpha_n^0 x_n + \sum_{k=1}^N \alpha_n^k T_k x_n$ for $n \in \mathbb{N}$. Indeed, if X is a closed convex subset of a Hilbert space with a norm $\|\cdot\|$, we have

$$\left\| \sum_{k=0}^{N} \alpha_k u_k \right\|^2 = \sum_{k=0}^{N} \alpha_k \|u_k\|^2 - \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \|u_i - u_j\|^2$$

for any $u_0, u_1, u_2, \dots, u_N \in H$ and $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_N \in [0,1]$ with $\sum_{k=0}^N \alpha_k = 1$. Therefore, we have

$$\begin{aligned} &\alpha_n^0 \|x_n - y\|^2 + \sum_{k=1}^{N} \alpha_n^k \|T_k x_n - y\|^2 \\ &= \left\| \alpha_n^0 x_n + \sum_{k=1}^{N} \alpha_n^k T_k x_n - y \right\|^2 \\ &\quad + \sum_{k=1}^{N} \alpha_n^0 \alpha_n^k \|x_n - T_k x_n\|^2 + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_n^i \alpha_n^j \|T_i x_n - T_j x_n\|^2, \end{aligned}$$

and hence its unique minimizer is $y = \alpha_n^0 x_n + \sum_{k=1}^N \alpha_n^k T_k x_n$.

The second result is a Δ-convergence theorem with an implicit iterative scheme for a finite family of nonexpansive mappings.

Theorem 3.2. *Let X be a Hadamard space. Let $T_k : X \to X$ be a nonexpansive mapping for $k = 1, 2, \dots, N$ and suppose that $\bigcap_{k=1}^N \operatorname{Fix} T_k \neq \emptyset$. Let $\{\alpha_n^k\}$ be real sequences for $k = 0, 1, 2, \dots, N$ and $a, b \in \mathbb{R}$ satisfying the following conditions:*

(i) $0 < a \le \alpha_n^k \le b < 1$ *for all* $k = 0, 1, 2, \dots, N$ *and* $n \in \mathbb{N}$;
(ii) $\sum_{k=0}^N \alpha_n^k = 1$ *for all* $n \in \mathbb{N}$.

For given $x_1 \in X$, generate a sequence $\{x_n\} \subset X$ as follows: For $n \in \mathbb{N}$ and given $x_n \in X$, let x_{n+1} be a unique point in X satisfying that

$$x_{n+1} = \operatorname*{argmin}_{y \in X} \left(\alpha_n^0 d(x_n, y)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_{n+1}, y)^2 \right).$$

Then, $\{x_n\}$ is well-defined and Δ-convergent to some $x_0 \in \bigcap_{k=1}^N \operatorname{Fix} T_k$.

Proof. We first show that $\{x_n\}$ is well-defined by induction. For $n = 1$, x_1 is a given point in X. Suppose $x_n \in \mathbb{N}$ is defined. Then define a mapping $V_n : X \to X$ by

$$V_n x = \operatorname*{argmin}_{y \in X} \left(\alpha_n^0 d(x_n, y)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x, y)^2 \right)$$

for $x \in X$. Notice that V_n can be defined as a single-valued mapping. Then V_n is a contraction. Indeed, for $u, v \in X$ and $t \in]0,1[$, we have

$$\begin{aligned}
&\alpha_n^0 d(x_n, V_n u)^2 + \sum_{k=1}^N \alpha_n^k d(T_k u, V_n u)^2 \\
&\le \alpha_n^0 d(x_n, tV_n u \oplus (1-t)V_n v)^2 + \sum_{k=1}^N \alpha_n^k d(T_k u, tV_n u \oplus (1-t)V_n v)^2 \\
&\le \alpha_n^0 \left(td(x_n, V_n u)^2 + (1-t)d(x_n, V_n v)^2 - t(1-t)d(V_n u, V_n v)^2\right) \\
&\quad + \sum_{k=1}^N \alpha_n^k \left(td(T_k u, V_n u)^2 + (1-t)d(T_k u, V_n v)^2 - t(1-t)d(V_n u, V_n v)^2\right) \\
&\le t\left(\alpha_n^0 d(x_n, V_n u)^2 + \sum_{k=1}^N \alpha_n^k d(T_k u, V_n u)^2\right) \\
&\quad + (1-t)\left(\alpha_n^0 d(x_n, V_n v)^2 + \sum_{k=1}^N \alpha_n^k d(T_k u, V_n v)^2\right) - t(1-t)d(V_n u, V_n v)^2,
\end{aligned}$$

and thus,

$$\begin{aligned}
t(1-t)d(V_n u, V_n v)^2 &\le (1-t)\alpha_n^0 \left(d(x_n, V_n v)^2 - d(x_n, V_n u)^2\right) \\
&\quad + (1-t)\sum_{k=1}^N \alpha_n^k \left(d(T_k u, V_n v)^2 - d(T_k u, V_n u)^2\right).
\end{aligned}$$

Dividing both sides by $1-t$ and tending $t \to 1$, we get

$$\begin{aligned}
d(V_n u, V_n v)^2 &\le \alpha_n^0 \left(d(x_n, V_n v)^2 - d(x_n, V_n u)^2\right) \\
&\quad + \sum_{k=1}^N \alpha_n^k \left(d(T_k u, V_n u)^2 - d(T_k u, V_n v)^2\right).
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
d(V_n v, V_n u)^2 &\le \alpha_n^0 \left(d(x_n, V_n u)^2 - d(x_n, V_n v)^2\right) \\
&\quad + \sum_{k=1}^N \alpha_n^k \left(d(T_k v, V_n v)^2 - d(T_k v, V_n u)^2\right).
\end{aligned}$$

From these inequalities, we get

$$\begin{aligned}
&2d(V_n u, V_n v)^2 \\
&\leq \sum_{k=1}^{N} \alpha_n^k \left(d(T_k u, V_n u)^2 + d(T_k v, V_n v)^2 - d(T_k u, V_n v)^2 - d(T_k v, V_n u)^2 \right) \\
&\leq \sum_{k=1}^{N} 2\alpha_n^k d(T_k u, T_k v) d(V_n u, V_n v) \\
&\leq 2 \sum_{k=1}^{N} \alpha_n^k d(u, v) d(V_n u, V_n v) \\
&\leq 2(1 - \alpha_n^0) d(u, v) d(V_n u, V_n v),
\end{aligned}$$

and hence

$$d(V_n u, V_n v) \leq (1 - \alpha_n^0) d(u, v).$$

Since $0 < 1 - \alpha_n^0 < 1$, V_n is a contraction and thus it has a unique fixed point $x_{n+1} \in X$. That is, it satisfies that

$$x_{n+1} = V_n x_{n+1} = \operatorname*{argmin}_{y \in X} \left(\alpha_n^0 d(x_n, y)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_{n+1}, y)^2 \right).$$

This implies that x_{n+1} satisfying this equation exists uniquely, and hence $\{x_n\}$ is well-defined.

For $p \in \bigcap_{k=1}^{N} \operatorname{Fix} T_k$, we have

$$\begin{aligned}
&\alpha_n^0 d(x_n, x_{n+1})^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, x_{n+1})^2 \\
&= \alpha_n^0 d(x_n, V_n x_{n+1})^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, V_n x_{n+1})^2 \\
&\leq \alpha_n^0 d(x_n, t x_{n+1} \oplus (1-t)p)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, t x_{n+1} \oplus (1-t)p)^2 \\
&\leq \alpha_n^0 \left(t d(x_n, x_{n+1})^2 + (1-t) d(x_n, p)^2 - t(1-t) d(x_{n+1}, p)^2 \right) \\
&\quad + \sum_{k=1}^{N} \alpha_n^k \left(t d(T_k x_n, x_{n+1})^2 + (1-t) d(T_k x_n, p)^2 - t(1-t) d(x_{n+1}, p)^2 \right) \\
&\leq \alpha_n^0 \left(t d(x_n, x_{n+1})^2 + (1-t) d(x_n, p)^2 - t(1-t) d(x_{n+1}, p)^2 \right) \\
&\quad + \sum_{k=1}^{N} \alpha_n^k \left(t d(T_k x_n, x_{n+1})^2 + (1-t) d(x_n, p)^2 - t(1-t) d(x_{n+1}, p)^2 \right) \\
&\leq t \left(\alpha_n^0 d(x_n, x_{n+1})^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, x_{n+1})^2 \right) \\
&\quad + (1-t) d(x_n, p)^2 - t(1-t) d(x_{n+1}, p)^2,
\end{aligned}$$

and hence

$$\alpha_n^0 d(x_n, x_{n+1})^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, x_{n+1})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

It follows that the nonnegative real sequence $\{d(x_n, p)^2\}$ is nonincreasing and thus it has a limit $c_p \in \mathbb{R}$. Then we get

$$\alpha_n^0 \lim_{n\to\infty} d(x_n, x_{n+1})^2 + \sum_{k=1}^{N} \lim_{n\to\infty} \alpha_n^k d(T_k x_n, x_{n+1})^2 \leq c_p^2 - c_p^2 = 0.$$

Since $\{\alpha_n^0\} \subset [a, b]$ and $\{\alpha_n^k\} \subset [a, b]$ for $k = 1, 2, \ldots, N$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(T_k x_n, x_{n+1}) = 0,$$

and hence

$$\lim_{n\to\infty} d(x_n, T_k x_n) = 0$$

for $k = 1, 2, \ldots, N$.

The remainder of the proof is the same as that of Theorem 3.1 and we finally get $\{x_n\}$ is Δ-convergent to $x_0 \in \bigcap_{k=1}^{N} \operatorname{Fix} T_k$. ■

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