



SOLUTIONS OF NONLINEAR TIME-FRACTIONAL WAVE-LIKE EQUATIONS WITH VARIABLE COEFFICIENTS IN THE FORM OF MITTAG-LEFFLER FUNCTIONS

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Abstract The purpose of this article is to propose a new modified method called fractional Elzaki projected differential transform method (FEPDTM) which is a combination of two powerful methods, the Elzaki transform method and projected differential transform method for giving the solutions of nonlinear wave-like differential equations with variable coefficients in the form of Mittag-leffler functions. The fractional derivative is described in the Caputo sense. Three numerical examples are presented to illustrate the applicability and the easiness of the proposed method.

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1. INTRODUCTION

Fractional partial differential equations obtained from standard partial differential equations by replacing the ordinary order derivative by a fractional derivative of order $\alpha > 0$, are used to model many mathematical and physical problems, such as fluid flow, finance, physical, hydrology, biological processes and systems and so on [7, 8, 10–13].

The most frequent used methods for investigating fractional partial differential equations are: Adomian decomposition method (ADM) [14] fractional variational iteration method (FVIM) [15], fractional difference method (FDM) [12], generalized differential transform method (GDTM) [1], homotopy analysis method (HAM) [2], homotopy perturbation method (HPM) [4], fractional residual power series method (FPSM) [6], modified generalized Taylor fractional series method (MGTFSM) [7].

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The main objective of this article is to present a new modified method called fractional Elzaki projected differential transform method (FEPDTM) for finding the solutions of nonlinear time-fractional wave-like equations with variable coefficients of the form

$$D_t^\alpha u = \sum_{i,j=1}^n F_{1ij}(X, t, u) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j}) + \sum_{i=1}^n G_{1i}(X, t, u) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{x_i}) + H(X, t, u) + S(X, t), \quad (1.1)$$

with the initial conditions

$$u(X, 0) = a_0(X), u_t(X, 0) = a_1(X), \quad (1.2)$$

where D_t^α is the Caputo fractional derivative operator of order α , $1 < \alpha \leq 2$.

Here $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, F_{1ij}, G_{1i} $i, j \in \{1, 2, \dots, n\}$ are nonlinear functions of X, t and u , F_{2ij}, G_{2i} $i, j \in \{1, 2, \dots, n\}$, are nonlinear functions of derivatives of u with respect to x_i and x_j $i, j \in \{1, 2, \dots, n\}$, respectively. Also H, S are nonlinear functions and k, m, p are integers.

For $\alpha = 2$, these types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows.

This article is organized as follows. In Section 2, we present some fundamental definitions of fractional calculus and Elzaki transform. In Section 3, we introduce the methodology of the fractional Elzaki projected differential transform method (FEPDTM) for solving nonlinear time-fractional wave-like equations with variable coefficients (1.1) with the initial conditions (1.2). In Section 4, we have proposed three examples to solve in order to show the validity and effectiveness of this approach. Finally, we present our obtained results (Graphs and Tables), comparing them with their exact associated forms. These results were verified with Matlab.

2. DEFINITION AND PRELIMINARIES

In this section, we give some definitions and important properties of the fractional calculus theory and Elzaki transform which shall be used in this paper.

2.1. FRACTIONAL CALCULUS

Definition 2.1. [8] A real function $f(t), t > 0$, is considered to be in the space C_μ , $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $f(t) = t^p h(t)$, where $h(t) \in C([0, \infty[)$, and it is said to be in the space C_μ^n if $f^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

Definition 2.2. [8] The Riemann-Liouville fractional integral operator I^α of order $\alpha \geq 0$ for a function $f \in C_\mu, \mu \geq -1$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi) d\xi, t > 0. \quad (2.1)$$

Definition 2.3. [8] The Caputo fractional derivative operator of order $n - 1 < \alpha \leq n$ for a function $f \in C_{-1}^n$ is defined as

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, t > 0. \tag{2.2}$$

For the Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, t > 0. \tag{2.3}$$

Definition 2.4. [8] The Mittag-Leffler function is defined as follows

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha + 1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{2.4}$$

A further generalization of Eq. (2.4) is given in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha + \beta)}, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0. \tag{2.5}$$

For $\alpha = 1$, $E_\alpha(z)$ reduces to e^z .

2.2. ELZAKI TRANSFORM

Recently, Tarig Elzaki [3] introduced a new integral transform, called Elzaki transform, which is applied to solve an ordinary and partial differential equations.

Definition 2.5. [3] The Elzaki transform is defined over the set of functions

$$A = \left\{ f(t) / \exists M, k_1, k_2 > 0, |f(t)| < M e^{\frac{|t|}{k_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}, \tag{2.6}$$

by the following integral

$$\mathcal{E}[f(t)] = T(v) = v \int_0^\infty f(t) e^{-\frac{t}{v}} dt, t > 0, \tag{2.7}$$

where v is the factor of variable t .

Some basic properties of the Elzaki transform are given as follows

Property 1: The Elzaki transform is a linear operator. That is, if λ and μ are non-zero constants, then

$$\mathcal{E}[\lambda f(t) \pm \mu g(t)] = \lambda \mathcal{E}[f(t)] \pm \mu \mathcal{E}[g(t)]. \tag{2.8}$$

Property 2: If $f^{(n)}(t)$ is the n -th derivative of the function $f(t) \in A$ with respect to " t " then its Elzaki transform is given by

$$\mathcal{E}[f^{(n)}(t)] = \frac{1}{v^n} T(v) - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0). \tag{2.9}$$

Property 3: (Convolution property) Suppose $T(v)$ and $G(v)$ are the Elzaki transforms of $f(t)$ and $g(t)$, respectively, both defined in the set A . Then the Elzaki transform of their convolution is given by

$$\mathcal{E}[(f * g)(t)] = \frac{1}{v}T(v)G(v), \quad (2.10)$$

where the convolution of two functions is defined by

$$(f * g)(t) = \int_0^t f(\xi)g(t - \xi)d\xi = \int_0^t f(t - \xi)g(\xi)d\xi. \quad (2.11)$$

Property 4: Some special Elzaki transforms:

$$\mathcal{E}(1) = v^2, \mathcal{E}(t) = v^3, \mathcal{E}(t^n) = n!v^{n+2}, n = 0, 1, 2, \dots \quad (2.12)$$

Property 5: The Elzaki transform of t^α is given by

$$\mathcal{E}[t^\alpha] = v^{\alpha+2}\Gamma(\alpha + 1). \quad (2.13)$$

2.3. ELZAKI TRANSFORM FOR FRACTIONAL DERIVATIVE

Theorem 2.6. If $T(v)$ is the Elzaki transform of $f(t)$, then the Elzaki transform of the Riemann-Liouville fractional integral for the function $f(t)$ of order α , is given by

$$\mathcal{E}[I^\alpha f(t)] = v^\alpha T(v). \quad (2.14)$$

Proof. The Riemann-Liouville fractional integral for the function $f(t)$, as in Eq. (2.1), can be expressed as the convolution

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t). \quad (2.15)$$

Applying the Elzaki transform in the Eq. (2.15) and using the Properties 3 and 5, we have

$$\mathcal{E}[I^\alpha f(t)] = \mathcal{E}\left[\frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t)\right] = v\mathcal{E}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\mathcal{E}[f(t)] = \frac{1}{v}v^{\alpha+1}T(v) = v^\alpha T(v). \quad (2.16)$$

The proof is complete. ■

Theorem 2.7. Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n - 1 < \alpha \leq n$ and $T(v)$ be the Elzaki transform of the function $f(t)$, then the Elzaki transform denoted by $T_\alpha(v)$ of the Caputo fractional derivative of $f(t)$ of order α , is given by

$$\mathcal{E}[D^\alpha f(t)] = T_\alpha(v) = \frac{1}{v^\alpha}T(v) - \sum_{k=0}^{n-1} v^{2-\alpha+k} f^{(k)}(0). \quad (2.17)$$

Proof. Let

$$g(t) = f^{(n)}(t), \quad (2.18)$$

then by the Definition of the Caputo fractional derivative 2.3, we obtain

$$\begin{aligned}
 D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi \\
 &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} g(\xi) d\xi \\
 &= I^{n-\alpha} g(t).
 \end{aligned}
 \tag{2.19}$$

Applying the Elzaki transform on both sides of Eq. (2.19) and using the Theorem 2.6, we get

$$\mathcal{E} [D^\alpha f(t)] = \mathcal{E} [I^{n-\alpha} g(t)] = v^{n-\alpha} G(v).
 \tag{2.20}$$

Also, we have from the Property 2

$$\mathcal{E} [g(t)] = \mathcal{E} [f^{(n)}(t)],
 \tag{2.21}$$

$$G(v) = \frac{1}{v^n} T(v) - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0).
 \tag{2.22}$$

Hence, (2.20) becomes

$$\begin{aligned}
 \mathcal{E} [D^\alpha f(t)] &= v^{n-\alpha} \left(\frac{1}{v^n} T(v) - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0) \right) \\
 &= \frac{1}{v^\alpha} T(v) - \sum_{k=0}^{n-1} v^{2-\alpha+k} f^{(k)}(0) = T_\alpha(v).
 \end{aligned}
 \tag{2.23}$$

The proof is complete. ■

3. FEPDTM FOR NONLINEAR TIME-FRACTIONAL WAVE-LIKE EQUATIONS

Theorem 3.1. Consider the following nonlinear time-fractional wave-like equations (1.1) with the initial conditions (1.2). Then, by FEPDTM the solution of Eqs. (1.1) and (1.2) is given in the form of infinite series which converges rapidly to the exact solution as follows

$$u(X, t) = \sum_{k=0}^{\infty} U(X, k),
 \tag{3.1}$$

where $U(X, k)$ is the projected differential transformed function.

Proof. In order to achieve our goal, we consider the following nonlinear time-fractional wave-like equations (1.1) with the initial conditions (1.2).

Applying the Elzaki transform on both sides of Eq. (1.1) and using the Theorem 2.7, we get

$$\begin{aligned} \mathcal{E}[u(X, t)] &= v^\alpha \sum_{k=0}^{n-1} v^{2-\alpha+k} [D^k u(X, t)]_{t=0} + v^\alpha \mathcal{E}[S(X, t)] \\ &+ v^\alpha \mathcal{E} \left[\sum_{i,j=1}^n F_{1ij}(X, t, u) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j}) \right. \\ &\left. + \sum_{i=1}^n G_{1i}(X, t, u) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{x_i}) + H(X, t, u) \right]. \end{aligned} \quad (3.2)$$

Taking the inverse Elzaki transform on both sides of Eq. (3.2), we have

$$\begin{aligned} u(X, t) &= L(X, t) + \mathcal{E}^{-1} \left(v^\alpha \mathcal{E} \left[\sum_{i,j=1}^n F_{1ij}(X, t, u) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j}) \right. \right. \\ &\left. \left. + \sum_{i=1}^n G_{1i}(X, t, u) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{x_i}) + H(X, t, u) \right] \right), \end{aligned} \quad (3.3)$$

where $L(X, t)$ is a term arising from the source term and the prescribed initial conditions.

Now, we apply the projected differential transform method (PDTM) [9] to Eq. (3.3), we get

$$\begin{aligned} U(X, 0) &= L(X, t), \\ U(X, k+1) &= \mathcal{E}^{-1} (v^\alpha \mathcal{E} [A(X, k) + B(X, k) + C(X, k)]), \quad k \geq 0, \end{aligned} \quad (3.4)$$

where $A(X, k)$, $B(X, k)$ and $C(X, k)$ are transformed form of the nonlinear terms,

$\sum_{i,j=1}^n F_{1ij}(X, t, u) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j})$, $\sum_{i=1}^n G_{1i}(X, t, u) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{x_i})$ and $H(X, t, u)$, respectively.

From Eq. (3.4) we have

$$\begin{aligned} U(X, 0) &= L(X, t), \\ U(X, 1) &= \mathcal{E}^{-1} (v^\alpha \mathcal{E} [A(X, 0) + B(X, 0) + C(X, 0)]), \\ U(X, 2) &= \mathcal{E}^{-1} (v^\alpha \mathcal{E} [A(X, 1) + B(X, 1) + C(X, 1)]), \\ U(X, 3) &= \mathcal{E}^{-1} (v^\alpha \mathcal{E} [A(X, 2) + B(X, 2) + C(X, 2)]), \\ &\vdots \end{aligned}$$

and so on.

Then, the solution of Eqs. (1.1) and (1.2) is given in the form of infinite series as follows

$$u(X, t) = \sum_{k=0}^{\infty} U(X, k). \quad (3.5)$$

The proof is complete. ■

4. APPLICATIONS

In this section, we apply the FEPDTM to solve three numerical examples of nonlinear time-fractional wave-like equations with variable coefficients.

Example 4.1. Let's consider the following two dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^\alpha u = \frac{\partial^2}{\partial x \partial y} (u_{xx} u_{yy}) - \frac{\partial^2}{\partial x \partial y} (xy u_x u_y) - u, 1 < \alpha \leq 2, \tag{4.1}$$

with the initial conditions

$$u(x, y, 0) = e^{xy}, u_t(x, y, 0) = e^{xy}, x, y \in \mathbb{R} \times \mathbb{R}. \tag{4.2}$$

By applying the steps involved in the FEPDTM as presented in Section 3 to Eqs. (4.1) and (4.2), we have

$$u(x, y, t) = e^{xy} + t e^{xy} + \mathcal{E}^{-1} \left[v^\alpha \mathcal{E} \left[\frac{\partial^2}{\partial x \partial y} (u_{xx} u_{yy}) - \frac{\partial^2}{\partial x \partial y} (xy u_x u_y) - u \right] \right]. \tag{4.3}$$

According to the PDTM, we can construct the following iteration formula

$$\begin{aligned} U(x, y, 0) &= e^{xy} + t e^{xy}, \tag{4.4} \\ U(x, y, k + 1) &= \mathcal{E}^{-1} \left(v^\alpha \mathcal{E} \left[\frac{\partial^2}{\partial x \partial y} A(x, y, k) - \frac{\partial^2}{\partial x \partial y} B(x, y, k) - U(x, y, k) \right] \right), k \geq 0, \end{aligned}$$

where $A(x, y, k)$ and $B(x, y, k)$ are transformed form of the nonlinear terms, $u_{xx} u_{yy}$ and $xy u_x u_y$, respectively, having the value

$$A(x, y, k) = \sum_{r=0}^k \frac{\partial^2 U(x, y, r)}{\partial x^2} \frac{\partial^2 U(x, y, k - r)}{\partial y^2}, \tag{4.5}$$

$$B(x, y, k) = xy \sum_{r=0}^k \frac{\partial U(x, y, r)}{\partial x} \frac{\partial U(x, y, k - r)}{\partial y}. \tag{4.6}$$

The components of $A(x, y, k)$ and $B(x, y, k)$ can be calculated by using (4.5) and (4.6), respectively.

From the relationship in (4.4), we obtain

$$\begin{aligned} u(x, y, 0) &= (1 + t) e^{xy}, \\ u(x, y, 1) &= - \left(\frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \right) e^{xy}, \\ u(x, y, 2) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \right) e^{xy}, \\ &\vdots \end{aligned}$$

Then, the approximate series solution of Eqs. (4.1) and (4.2) can be expressed by

$$\begin{aligned} u(x, y, t) &= \left(1 + t - \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - \dots \right) e^{xy} \\ &= (E_\alpha(-t^\alpha) + t E_{\alpha,2}(-t^\alpha)) e^{xy}, \tag{4.7} \end{aligned}$$

where $E_\alpha(-t^\alpha)$ and $E_{\alpha,2}(-t^\alpha)$ are the Mittag-Leffler functions defined by Eqs. (2.4) and (2.5).

Taking $\alpha = 2$ in Eq. (4.7), the approximate series solution of Eqs. (4.1) and (4.2) has the general pattern form which is coinciding with the following exact solution in terms of infinite series

$$u(x, y, t) = \left(1 + t - \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \dots \right) e^{xy}. \tag{4.8}$$

So, the exact solution of Eqs. (4.1) and (4.2) in a closed form of elementary function will be

$$u(x, y, t) = (\cos t + \sin t) e^{xy}. \tag{4.9}$$

The above expressions is exactly same as those given by the FRPSM [6].

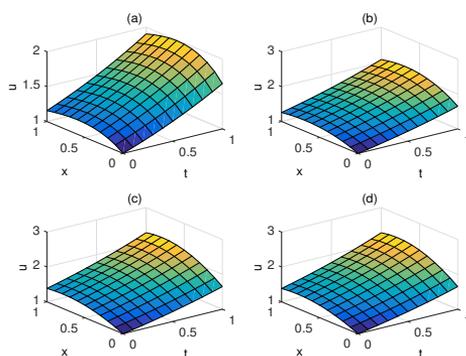


FIGURE 1. The surface graph of the 3-term approximate solutions and exact solution for Example 4.1 when $y = 0.5$: (a) u when $\alpha = 1.5$, (b) u when $\alpha = 1.75$, (c) u when $\alpha = 2$, and (d) u exact.

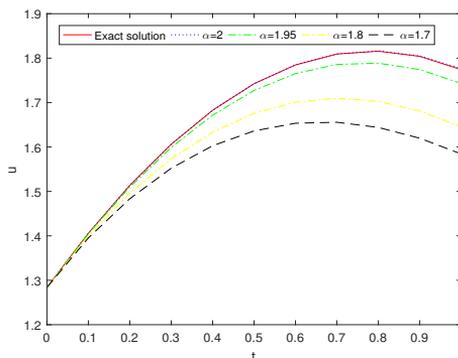


FIGURE 2. The behavior of the exact solution and 3-term approximate solutions for different values of α for Example 4.1 when $x = y = 0.5$.

$t/x, y$	0.1	0.3	0.5	0.7
0.1	1.4226×10^{-9}	1.5411×10^{-9}	1.8085×10^{-9}	2.2991×10^{-9}
0.3	1.0648×10^{-6}	1.1535×10^{-6}	1.3536×10^{-6}	1.7208×10^{-6}
0.5	2.3382×10^{-5}	2.5330×10^{-5}	2.9725×10^{-5}	3.7787×10^{-5}
0.7	1.8000×10^{-4}	1.9499×10^{-4}	2.2882×10^{-4}	2.9089×10^{-4}
0.9	8.2963×10^{-4}	8.9872×10^{-4}	1.0547×10^{-3}	1.3407×10^{-3}

TABLE 1. Comparison of the absolute errors for the obtained results and exact solution for Example 4.1, when $n = 3$ and $\alpha = 2$.

Example 4.2. Let’s consider the following one dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^\alpha u = u^2 \frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^2 \frac{\partial^2}{\partial x^2}(u_{xx}^3) - 18u^5 + u, 1 < \alpha \leq 2, \tag{4.10}$$

with the initial conditions

$$u(x, 0) = e^x, u_t(x, 0) = e^x, x \in]0, 1[. \tag{4.11}$$

By applying the steps involved in the FEPDTM as presented in Section 3 to Eqs. (4.10) and (4.11), we have

$$u(x, t) = e^x + te^x + \mathcal{E}^{-1} \left[v^\alpha \mathcal{E} \left[u^2 \frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^2 \frac{\partial^2}{\partial x^2}(u_{xx}^3) - 18u^5 + u \right] \right].$$

According to the PDTM, we can construct the following iteration formula

$$\begin{aligned} U(x, 0) &= (1 + t) e^x, \\ U(x, k + 1) &= \mathcal{E}^{-1} [v^\alpha \mathcal{E} [A(x, k) + B(x, k) - 18C(x, k) + U(x, k)]] , k \geq 0, \end{aligned} \tag{4.12}$$

where $A(x, k)$, $B(x, k)$ and $C(x, k)$ are transformed form of the nonlinear terms,

$u^2 \frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx})$, $u_x^2 \frac{\partial^2}{\partial x^2}(u_{xx}^3)$ and u^5 , respectively, having the value

$$\begin{aligned} A(x, k) &= \sum_{r=0}^k \sum_{s=0}^r \sum_{m=0}^s \sum_{n=0}^m U(x, n)U(x, m - n) \\ &\quad \times \frac{\partial^2}{\partial x^2} \left[\frac{\partial U(x, s - m)}{\partial x} \frac{\partial^2 U(x, r - s)}{\partial x^2} \frac{\partial^3 U(x, k - r)}{\partial x^3} \right], \end{aligned} \tag{4.13}$$

$$\begin{aligned} B(x, k) &= \sum_{r=0}^k \sum_{s=0}^r \sum_{m=0}^s \sum_{n=0}^m \frac{\partial U(x, n)}{\partial x} \frac{\partial U(x, m - n)}{\partial x} \\ &\quad \times \frac{\partial^2}{\partial x^2} \left[\frac{\partial^2 U(x, s - m)}{\partial x^2} \frac{\partial^2 U(x, r - s)}{\partial x^2} \frac{\partial^2 U(x, k - r)}{\partial x^2} \right], \end{aligned} \tag{4.14}$$

$$C(x, k) = \sum_{r=0}^k \sum_{s=0}^r \sum_{m=0}^s \sum_{n=0}^m U(x, n)U(x, m - n)U(x, s - m)U(x, r - s)U(x, k - r). \tag{4.15}$$

The components of $A(x, k)$, $B(x, k)$ and $C(x, k)$ can be calculated by using (4.13), (4.14) and (4.15), respectively.

From the relationship in (4.12), we obtain

$$\begin{aligned} U(x, 0) &= (1+t)e^x, \\ U(x, 1) &= \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right) e^x, \\ U(x, 2) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \right) e^x, \\ &\vdots \end{aligned}$$

Then, the approximate series solution of Eqs. (4.10) and (4.11) can be expressed by

$$\begin{aligned} u(x, t) &= \left(1+t + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \dots \right) e^x \\ &= (E_\alpha(t^\alpha) + tE_{\alpha,2}(t^\alpha)) e^x, \end{aligned} \quad (4.16)$$

where $E_\alpha(t^\alpha)$ and $E_{\alpha,2}(t^\alpha)$ are the Mittag-Leffler functions, defined by Eqs. (2.4) and (2.5).

Taking $\alpha = 2$ in Eq. (4.16), the approximate series solution of Eqs. (4.10) and (4.11) has the general pattern form which is coinciding with the following exact solution in terms of infinite series

$$u(x, t) = \left(1+t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \dots \right) e^x. \quad (4.17)$$

So, the exact solution of Eqs. (4.10) and (4.11) in a closed form of elementary function will be

$$u(x, t) = e^{x+t}. \quad (4.18)$$

The above expressions is exactly same as those given by the FRPSM [6].

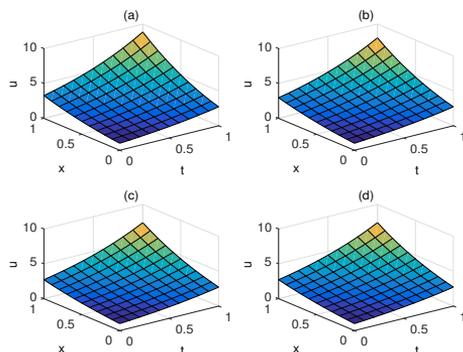


FIGURE 3. The surface graph of the 3-term approximate solutions and exact solution for Example 4.2: (a) u when $\alpha = 1.5$, (b) u when $\alpha = 1.75$, (c) u when $\alpha = 2$, and (d) u exact.

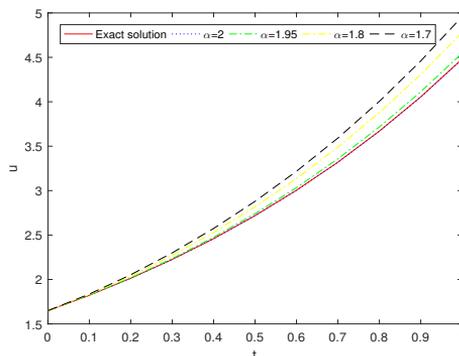


FIGURE 4. The behavior of the exact solution and 3-term approximate solutions for different values of α for Example 4.2 when $x = 0.5$.

$t/x, y$	0.1	0.3	0.5	0.7
0.1	1.5572×10^{-9}	1.9019×10^{-9}	2.3230×10^{-9}	2.8373×10^{-9}
0.3	1.1688×10^{-6}	1.4276×10^{-6}	1.7436×10^{-6}	2.1297×10^{-6}
0.5	2.5810×10^{-5}	3.1525×10^{-5}	3.8504×10^{-5}	4.7029×10^{-5}
0.7	2.0036×10^{-4}	2.4472×10^{-4}	2.9890×10^{-4}	3.6507×10^{-4}
0.9	9.3372×10^{-4}	1.1404×10^{-3}	1.3929×10^{-3}	1.7013×10^{-3}

TABLE 2. Comparison of the absolute errors for the obtained results and exact solution for Example 4.2, when $n = 3$ and $\alpha = 2$.

Example 4.3. Let’s consider the following one dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^\alpha u = x^2 \frac{\partial}{\partial x} (u_x u_{xx}) - x^2 (u_{xx}^2) - u, \quad 1 < \alpha \leq 2, \tag{4.19}$$

with the initial conditions

$$u(x, 0) = 0, u_t(x, 0) = x^2, x \in]0, 1[. \tag{4.20}$$

By applying the steps involved in the FEPDTM as presented in Section 3 to Eqs. (4.19) and (4.20), we have

$$u(x, t) = tx^2 + \mathcal{E}^{-1} \left[v^\alpha \mathcal{E} \left[x^2 \frac{\partial}{\partial x} (u_x u_{xx}) - x^2 (u_{xx}^2) - u \right] \right]. \tag{4.21}$$

According to the PDTM, we can construct the following iteration formula

$$\begin{aligned} U(x, 0) &= tx^2, \\ U(x, k + 1) &= \mathcal{E}^{-1} \left[v^\alpha \mathcal{E} \left[x^2 \frac{\partial}{\partial x} A(x, k) - x^2 B(x, k) - U(x, k) \right] \right], k \geq 0, \end{aligned} \tag{4.22}$$

where $A(x, k)$ and $B(x, k)$ are transformed form of the nonlinear terms, $u_x u_{xx}$ and u_{xx}^2 , respectively, having the value

$$A(x, k) = \sum_{r=0}^k \frac{\partial U(x, r)}{\partial x} \frac{\partial^2 U(x, k-r)}{\partial x^2}, \tag{4.23}$$

$$B(x, k) = \sum_{r=0}^k \frac{\partial^2 U(x, r)}{\partial x^2} \frac{\partial^2 U(x, k-r)}{\partial x^2}, \tag{4.24}$$

The components of $A(x, k)$ and $B(x, k)$ can be calculated by using (4.23) and (4.24), respectively.

From the relationship in (4.22), we obtain

$$\begin{aligned} U(x, 0) &= tx^2, \\ U(x, 1) &= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2, \\ U(x, 2) &= \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2, \\ &\vdots \end{aligned}$$

Then, the approximate series solution of Eqs. (4.20) and (4.22) can be expressed by

$$\begin{aligned} u(x, t) &= x^2 \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \dots \right) \\ &= x^2 (tE_{\alpha,2}(-t^\alpha)), \end{aligned} \tag{4.25}$$

where $E_{\alpha,2}(-t^\alpha)$ is the Mittag-Leffler function, defined by Eq. (2.4).

Taking $\alpha = 2$ in Eq. (4.25), the approximate series solution of Eqs. (4.19) and (4.20) has the general pattern form which is coinciding with the following exact solution in terms of infinite series

$$u(x, t) = x^2 \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \right). \tag{4.26}$$

So, the exact solution of Eqs. (4.19) and (4.20) in a closed form of elementary function will be

$$u(x, t) = x^2 \sin t. \tag{4.27}$$

The above expressions is exactly same as those given by the FRPSM [6].

t/x	0.1	0.3	0.5	0.7
0.1	1.9839×10^{-13}	1.7855×10^{-12}	4.9596×10^{-12}	9.7209×10^{-12}
0.3	4.3339×10^{-10}	3.9005×10^{-9}	1.0835×10^{-8}	2.1236×10^{-8}
0.5	1.5447×10^{-8}	1.3903×10^{-7}	3.8618×10^{-7}	7.5692×10^{-7}
0.7	1.6229×10^{-7}	1.4606×10^{-6}	4.0574×10^{-6}	7.9524×10^{-6}
0.9	9.3840×10^{-7}	8.4456×10^{-6}	2.3460×10^{-5}	4.5982×10^{-5}

TABLE 3. Comparison of the absolute errors for the obtained results and exact solution for Example 4.3, when $n = 3$ and $\alpha = 2$.

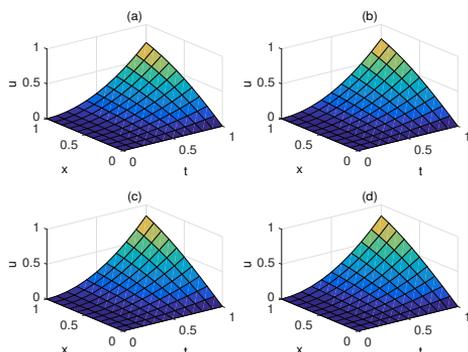


FIGURE 5. The surface graph of the 3-term approximate solutions and exact solution for Example 4.3: (a) u when $\alpha = 1.5$, (b) u when $\alpha = 1.75$, (c) u when $\alpha = 2$, and (d) u exact.

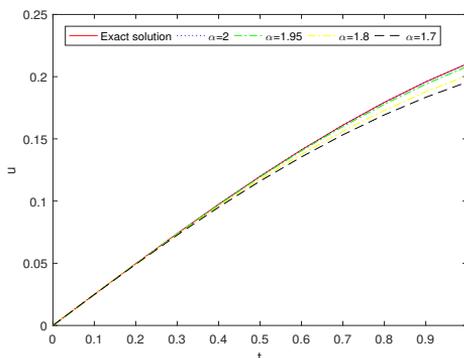


FIGURE 6. The behavior of the exact solution and 3-term approximate solutions for different values of α for Example 4.3 when $x = 0.5$.

Remark 4.4. The numerical results (See Figures 1–6, and Tables 1–3), affirm that when α approaches 2, our results obtained by the FEPDTM approach the exact solutions.

Remark 4.5. In this article, we only apply three terms to approximate the solutions, if we apply more terms of the approximate solutions, the accuracy of the approximate solutions will be greatly improved.

5. CONCLUSION

In this article, the fractional Elzaki projected differential transform method (FEPDTM) has been successfully applied to study the solutions of nonlinear time-fractional wave-like equations with variable coefficients. The results show that the FEPDTM is an efficient and easy to use technique for finding approximate series solutions for these equations. The obtained approximate series solutions using the suggested method is in excellent agreement with the exact solution. This confirms our belief that the efficiency of our

technique gives it much wider applicability for general classes of linear and nonlinear fractional partial differential equations.

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