



A SPHERICALLY VICINAL MAPPING ON GEODESIC SPACES WITH CURVATURE BOUNDED ABOVE

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Abstract In 2018, vicinal mappings and firmly vicinal mappings were proposed by Kohsaka. He showed fundamental properties of firmly vicinal mappings and a fixed point theorem for those mappings. In this paper, we propose spherically vicinal mappings and firmly spherically vicinal mappings whose classes contain those of vicinal mappings and firmly vicinal mappings, respectively. We also show some examples of its mapping and prove fundamental properties and an approximation theorem to a fixed point.

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1. INTRODUCTION

In 2018, Kohsaka [8] proposed vicinal mappings and firmly vicinal mappings. In an admissible CAT(1) space X , a mapping T from X into itself is said to be

- vicinal if

$$\begin{aligned} & (C_x^2(1 + C_y^2) + C_y^2(1 + C_x^2)) \cos d(Tx, Ty) \\ & \geq C_x^2(1 + C_y^2) \cos d(Tx, y) + C_y^2(1 + C_x^2) \cos d(x, Ty) \end{aligned}$$

for all $x, y \in X$;

- firmly vicinal if

$$\begin{aligned} & (C_x^2(1 + C_y^2)C_y + C_y^2(1 + C_x^2)C_x) \cos d(Tx, Ty) \\ & \geq C_x^2(1 + C_y^2) \cos d(Tx, y) + C_y^2(1 + C_x^2) \cos d(x, Ty) \end{aligned}$$

for all $x, y \in X$,

where $C_z = \cos d(Tz, z)$ for all $z \in X$. Since X is an admissible CAT(1) space, it is clear that every firmly vicinal mapping is vicinal mapping.

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In an admissible complete CAT(1) space X , two kinds of resolvents for a proper lower semicontinuous convex function f were proposed recently. In 2016, first one was proposed by Kimura and Kohsaka [5] as follows:

$$R_f x = \operatorname{argmin}_{y \in X} \{f(y) + \tan d(y, x) \sin d(y, x)\} \tag{1.1}$$

for each $x \in X$. They proved that it is well-defined as a single-valued mapping, and the following inequality which directly implies the firm vicinality of R_f holds:

$$\begin{aligned} & \left(\left(1 + \frac{1}{C_x^2}\right) C_x + \left(1 + \frac{1}{C_y^2}\right) C_y \right) \cos d(R_f x, R_f y) \\ & \geq \left(1 + \frac{1}{C_y^2}\right) \cos d(R_f x, y) \left(1 + \frac{1}{C_x^2}\right) \cos d(x, R_f y) \end{aligned}$$

for all $x, y \in X$.

In 2019, the authors [2] proposed the second one as follows:

$$Q_f x = \operatorname{argmin}_{y \in X} \{f(y) - \log(\cos d(y, x))\} \tag{1.2}$$

for each $x \in X$. They showed that it is well-defined as a single-valued mapping, and the following inequality holds by using the same technique as the proof of the previous result:

$$2 \cos d(Q_f x, Q_f y) \geq \frac{1}{C_y} \cos d(Q_f x, y) + \frac{1}{C_x} \cos d(x, Q_f y)$$

for all $x, y \in X$. Since X is an admissible CAT(1) space, it is obvious that this inequality implies the spherical nonspreadingness of sum-type, that is,

$$2 \cos d(Q_f x, Q_f y) \geq \cos d(Q_f x, y) + \cos d(x, Q_f y)$$

for all $x, y \in X$. However, it does not imply the firm vicinality.

In this paper, we propose new notions of spherically vicinal mappings and firmly spherically vicinal mappings whose classes contain those of original ones and show that both resolvents mentioned above are spherically firmly vicinal in the new sense. We also show fundamental properties and an approximation theorem to a fixed point for those mappings.

2. PRELIMINARIES

In this paper, we use the notations that \mathbb{R} is the set of all real numbers, X is a metric space with metric d , $\mathcal{F}(T)$ is the set of all fixed points of a mapping T from X into itself, and $\operatorname{argmin}_{x \in X} f(x)$ is the set of all minimizers of a function f from X into $]-\infty, \infty]$.

For each $x, y \in X$, a mapping γ_{xy} from X into $[0, \ell]$ is called a geodesic joining x and y if γ_{xy} satisfies $\gamma_{xy}(0) = x$, $\gamma_{xy}(\ell) = y$, and $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s - t|$ for all $s, t \in [0, \ell]$, where $\ell = d(x, y)$. A metric space X is called a geodesic space if for each $x, y \in X$, there exists a geodesic γ_{xy} . In general, a geodesic is not always unique. In this paper, we assume that it is unique. Let X be a geodesic space. For each $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in X$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$, and we denote it by $z = tx \oplus (1 - t)y$, which is called a convex combination between x and y . We define a triangle on X by the set $\operatorname{Im}\gamma_{xy} \cup \operatorname{Im}\gamma_{yz} \cup \operatorname{Im}\gamma_{zx}$ and denote it by $\Delta(x, y, z)$. For each $\Delta(x, y, z) \subset X$ satisfying $d(x, y) + d(y, z) + d(z, x) < 2\pi$, its comparison triangle on \mathbb{S}^2 is defined by the set $\operatorname{Im}\gamma_{\bar{x}\bar{y}} \cup \operatorname{Im}\gamma_{\bar{y}\bar{z}} \cup \operatorname{Im}\gamma_{\bar{z}\bar{x}}$ satisfying $d(x, y) = d(\bar{x}, \bar{y})$, $d(y, z) = d(\bar{y}, \bar{z})$ and $d(z, x) = d(\bar{z}, \bar{x})$, and denoted by $\Delta(\bar{x}, \bar{y}, \bar{z})$, where \mathbb{S}^2 is the two-dimensional unit sphere

in \mathbb{R}^3 . For each $p \in \Delta(x, y, z) \subset X$, a point $\bar{p} \in \Delta(\bar{x}, \bar{y}, \bar{z}) \subset \mathbb{S}^2$ is called a comparison point of p if the following conditions hold:

- if $p \in \gamma_{xy}$, then $\bar{p} \in \gamma_{\bar{x}\bar{y}}$ and $d(x, p) = d(\bar{x}, \bar{p})$;
- if $p \in \gamma_{yz}$, then $\bar{p} \in \gamma_{\bar{y}\bar{z}}$ and $d(y, p) = d(\bar{y}, \bar{p})$;
- if $p \in \gamma_{zx}$, then $\bar{p} \in \gamma_{\bar{z}\bar{x}}$ and $d(z, p) = d(\bar{z}, \bar{p})$.

A geodesic space X is called a CAT(1) space if for any $\Delta(x, y, z)$ satisfying $d(x, y) + d(y, z) + d(z, x) < 2\pi$, it holds that $d(p, q) \leq d(\bar{p}, \bar{q})$ whenever p and q are elements of $\Delta(x, y, z)$, and \bar{p} and \bar{q} are comparison points of p and q , respectively. A CAT(1) space X is said to be admissible if $d(x, y) < \pi/2$ for all $x, y \in X$. In CAT(1) spaces, the following inequality which is called the CN-inequality is well known.

Lemma 2.1 (Kimura and Satô [4]). *Let X be a CAT(1) space, $x, y, z \in X$ satisfying $d(x, y) + d(y, z) + d(z, x) < 2\pi$, and $t \in [0, 1]$. Then*

$$\begin{aligned} & \cos d(tx \oplus (1 - t)y, z) \sin d(x, y) \\ & \geq \cos d(x, z) \sin(td(x, y)) + \cos d(y, z) \sin((1 - t)d(x, y)). \end{aligned}$$

Let X be a metric space and $\{x_n\}$ a sequence of X . An asymptotic center of $\{x_n\}$ is defined by the set

$$\left\{ u \in X \mid \limsup_{n \rightarrow \infty} d(u, x_n) = \inf_{y \in X} \limsup_{n \rightarrow \infty} d(y, x_n) \right\},$$

and we denote it by $\mathcal{A}(\{x_n\})$. A sequence $\{x_n\}$ is Δ -convergent to $x_0 \in X$ if the asymptotic center of each subsequence of $\{x_n\}$ is $\{x_0\}$, and we denote it by $x_n \xrightarrow{\Delta} x_0$. Such x_0 is called a Δ -limit of $\{x_n\}$. The following lemmas are well known.

Lemma 2.2 (Espínola and Fernández-León [1]). *Let X be an admissible complete CAT(1) space and $\{x_n\}$ a sequence of X . If*

$$\inf_{y \in X} \limsup_{n \rightarrow \infty} d(x_n, y) < \frac{\pi}{2},$$

then the following properties hold:

- $\mathcal{A}(\{x_n\})$ consists of one point;
- $\{x_n\}$ has a Δ -convergent subsequence.

Lemma 2.3 (Kimura, Seajung, and Yotkaew [7]). *Let X be an admissible complete CAT(1) space and $\{x_n\}$ a sequence of X satisfying*

$$\inf_{y \in X} \limsup_{n \rightarrow \infty} d(x_n, y) < \frac{\pi}{2}.$$

If $\{d(x_n, z)\}$ is convergent for each Δ -limit z of subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then $\{x_n\}$ is Δ -convergent to an element of X .

A function f from X into $]-\infty, \infty]$ is said to be proper if $f(x) < \infty$ for some $x \in X$, and f is said to be lower semicontinuous if $f(x_0) \leq \liminf_{n \rightarrow \infty} f(x_n)$ whenever $\{x_n\}$ is convergent to $x_0 \in X$. It is well known that f is lower semicontinuous if and only if the set $\{x \in X \mid f(x) \leq a\}$ is closed for all $a \in \mathbb{R}$. Let X be a geodesic space. A function f is said to be convex if $f(tx \oplus (1 - t)y) \leq tf(x) + (1 - t)f(y)$ for all $x, y \in X$ and $t \in]0, 1]$. A subset C of X is said to be convex if $tx \oplus (1 - t)y \in C$ for all $x, y \in C$ and $t \in]0, 1]$.

3. SPHERICALLY VICINAL MAPPINGS AND FIRMLY SPHERICALLY VICINAL MAPPINGS

In this section, we propose spherically vicinal mappings and firmly spherically vicinal mappings, and we introduce some examples of those mappings. Throughout this paper, for a mapping T from X into itself, we use the notation that $C_z = \cos d(Tz, z)$ for all $z \in X$.

Let X be an admissible CAT(1) space, T a mapping from X into itself and φ a function from $]0, 1[$ into $[0, \infty[$ satisfying the following conditions:

- φ is differentiable;
- $\varphi'(t) < 0$ for all $t \in]0, 1[$;
- φ' is continuous at 1.

Then T is said to be

- spherically vicinal with φ if

$$\begin{aligned} &(\varphi'(C_x) + \varphi'(C_y)) \cos d(Tx, Ty) \\ &\leq \varphi'(C_y) \cos d(Tx, y) + \varphi'(C_x) \cos d(Ty, x) \end{aligned}$$

for all $x, y \in X$;

- firmly spherically vicinal with φ if

$$\begin{aligned} &(\varphi'(C_x)C_x + \varphi'(C_y)C_y) \cos d(Tx, Ty) \\ &\leq \varphi'(C_y) \cos d(Tx, y) + \varphi'(C_x) \cos d(Ty, x) \end{aligned}$$

for all $x, y \in X$.

Since X is an admissible CAT(1) space, every firmly spherically vicinal mapping with φ is spherically vicinal with φ .

First we show a theorem which plays an important role for finding some examples of firmly spherically vicinal mappings with φ .

Theorem 3.1. *Let X be an admissible CAT(1) space, f a convex function from X into $]-\infty, \infty]$ and $\varphi:]0, 1[\rightarrow [0, \infty[$ a function with decreasing and differentiable. If*

$$Tx = \operatorname{argmin}_{y \in X} \{f(y) + \varphi(\cos d(y, x))\}$$

is a single-valued mapping, then

$$\begin{aligned} &(\varphi'(C_x)C_x + \varphi'(C_y)C_y) \cos d(Tx, Ty) \\ &\leq \varphi'(C_y) \cos d(Tx, y) + \varphi'(C_x) \cos d(Ty, x) \end{aligned}$$

for all $x, y \in X$.

Proof. Let $x, y \in X$ satisfying $Tx \neq Ty$ and put $z_t = tTx \oplus (1 - t)Ty$ for all $t \in]0, 1[$. By the definition of T and convexity of f , we have

$$\begin{aligned} f(Ty) + \varphi(C_y) &\leq f(z_t) + \varphi(\cos d(z_t, y)) \\ &\leq tf(Tx) + (1 - t)f(Ty) + \varphi(\cos d(z_t, y)) \end{aligned}$$

and hence

$$t(f(Ty) - f(Tx)) \leq \varphi(\cos d(z_t, y)) - \varphi(C_y).$$

Putting $D = d(Tx, Ty)$ and multiplying $(\sin D)/t$, we get

$$(f(Ty) - f(Tx)) \sin D \leq \frac{\varphi(C_y + \Delta(t)) - \varphi(C_y)}{\Delta(t)} \times \frac{\Delta(t) \sin D}{t},$$

where $\Delta(t) = \cos d(z_t, y) - C_y$. It is clear that $\Delta(t) \rightarrow 0$ as $t \downarrow 0$. On the other hand, using CN-inequality, we have

$$\begin{aligned} \Delta(t) \sin D &= \cos d(z_t, y) \sin D - C_y \sin D \\ &\geq \cos d(Tx, y) \sin(tD) + C_y \sin((1 - t)D) - C_y \sin D \\ &= \cos d(Tx, y) \sin(tD) + C_y(\sin((1 - t)D) - \sin D). \end{aligned}$$

Since φ is decreasing, we know that

$$\frac{\varphi(C_y + \Delta(t)) - \varphi(C_y)}{\Delta(t)} < 0.$$

Therefore we have

$$\begin{aligned} &(f(Ty) - f(Tx)) \sin D \\ &\leq \frac{\varphi(C_y + \Delta(t)) - \varphi(C_y)}{\Delta(t)} \times \frac{\Delta(t) \sin D}{t} \\ &\leq \frac{\varphi(C_y + \Delta(t)) - \varphi(C_y)}{\Delta(t)} \left(\cos d(Tx, y) \frac{\sin(tD)}{t} + C_y \frac{\sin((1 - t)D) - \sin D}{t} \right). \end{aligned}$$

Taking the limit as $t \downarrow 0$, we get

$$(f(Ty) - f(Tx)) \sin D \leq \varphi'(C_y)(\cos d(Tx, y) - C_y \cos D)D.$$

Similarly it follows that

$$(f(Tx) - f(Ty)) \sin D \leq \varphi'(C_x)(\cos d(Ty, x) - C_x \cos D)D.$$

Adding each side of these inequalities and dividing by D , we have

$$0 \leq \varphi'(C_y)(\cos d(Tx, y) - C_y \cos D) + \varphi'(C_x)(\cos d(Ty, x) - C_x \cos D).$$

From this inequality, we directly get the conclusion. In the case that $Tx = Ty$, we clearly obtain the conclusion. ■

Next we show the following examples by using Theorem 3.1.

Example 3.2. Let X be an admissible complete CAT(1) space, f a proper lower semi-continuous convex function from X into $]-\infty, \infty]$ and R_f is the resolvent which is defined by (1.1). Then R_f is firmly spherically vicinal with $\varphi: t \mapsto 1/t - t$.

Proof. We define a function $\varphi:]0, 1] \rightarrow [0, \infty[$ as follows:

$$\varphi(t) = \frac{1}{t} - t.$$

Since $\tan a \sin a = 1/\cos a - \cos a$ for all $a \in [0, \pi/2[$, we can express

$$R_f x = \operatorname{argmin}_{y \in X} \{f(y) + \varphi(\cos d(y, x))\}$$

for each $x \in X$, and we know that R_f is a single-valued mapping. It is obvious that the function φ is differentiable and

$$\varphi'(t) = -\frac{1}{t^2} - 1 < 0$$

for all $t \in]0, 1]$. Therefore, using Theorem 3.1, we get

$$\begin{aligned} & (\varphi'(C_x)C_x + \varphi'(C_y)C_y) \cos d(R_fx, R_fy) \\ & \leq \varphi'(C_y) \cos d(R_fx, y) + \varphi'(C_x) \cos d(R_fy, x) \end{aligned}$$

for all $x, y \in X$. We also know that φ' is continuous at 1. Thus R_f is firmly spherically vicinal with $\varphi: t \mapsto 1/t - t$. ■

Example 3.3. Let X be an admissible complete CAT(1) space, f a proper lower semi-continuous convex function from X into $] -\infty, \infty]$ and Q_f is the resolvent which is defined by (1.2). Then Q_f is firmly spherically vicinal with $\varphi: t \mapsto -\log t$.

Proof. We define a function $\varphi:]0, 1] \rightarrow [0, \infty[$ as $\varphi(t) = -\log t$. Then we know that φ is differentiable and

$$\varphi'(t) = -\frac{1}{t} < 0$$

for all $t \in]0, 1]$. Since Q_f is a single-valued mapping, we have

$$\begin{aligned} & (\varphi'(C_x)C_x + \varphi'(C_y)C_y) \cos d(Q_fx, Q_fy) \\ & \leq \varphi'(C_y) \cos d(Q_fx, y) + \varphi'(C_x) \cos d(Q_fy, x) \end{aligned}$$

for all $x, y \in X$ by Theorem 3.1. We also know that φ' is continuous at 1. Thus Q_f is firmly vicinal with $\varphi: t \mapsto -\log t$. ■

Remark 3.4. Every firmly vicinal mapping in the sense of Kohsaka [8] on admissible CAT(1) spaces is firmly vicinal with $\varphi: t \mapsto 1/t - t$. In fact, if T is firmly vicinal in the sense of Kohsaka, then

$$\begin{aligned} & (C_x^2(1 + C_y^2)C_y + C_y^2(1 + C_x^2)C_x) \cos d(Tx, Ty) \\ & \geq C_x^2(1 + C_y^2) \cos d(Tx, y) + C_y^2(1 + C_x^2) \cos d(x, Ty). \end{aligned}$$

Dividing $-C_x^2C_y^2$, we get

$$\begin{aligned} & \left(\left(-\frac{1}{C_x^2} - 1 \right) C_x + \left(-\frac{1}{C_y^2} - 1 \right) C_y \right) \cos d(Tx, Ty) \\ & \leq \left(-\frac{1}{C_y^2} - 1 \right) \cos d(Tx, y) \left(-\frac{1}{C_x^2} - 1 \right) \cos d(x, Ty). \end{aligned}$$

Therefore T is firmly spherically vicinal with $\varphi: t \mapsto 1/t - t$.

4. FUNDAMENTAL PROPERTIES AND AN APPROXIMATION THEOREM FOR FIRMLY SPHERICALLY VICINAL MAPPINGS

In this section, we show fundamental properties and an approximation theorem for firmly spherically vicinal mappings with φ . We assume that φ is a function from $]0, 1]$ into $[0, \infty[$ satisfying that φ is differentiable, $\varphi'(t) < 0$ for all $t \in]0, 1]$ and φ' is continuous at 1.

Let X be a metric space and T a mapping from X into itself. Then T is said to be

- quasi-nonexpansive if $\mathcal{F}(T)$ is nonempty and $d(Tx, p) \leq d(x, p)$ for all $x \in X$ and $p \in \mathcal{F}(T)$;
- Δ -demiclosed if p is a fixed point whenever a sequence $\{x_n\}$ of X satisfies $x_n \xrightarrow{\Delta} p \in X$ and $d(Tx_n, x_n) \rightarrow 0$ as $n \rightarrow \infty$;

- asymptotically regular if $\lim_{n \rightarrow \infty} d(T^{n+1}x, T^n x) = 0$ for all $x \in X$.

Theorem 4.1. *Let X be an admissible CAT(1) space and T a spherically vicinal mapping with φ from X into itself. Then T is quasi-nonexpansive.*

Proof. Let $x \in X$ and $p \in \mathcal{F}(T)$. Then the spherical vicinality with φ implies that

$$(\varphi'(C_x) + \varphi'(1)) \cos d(Tx, p) \leq \varphi'(1) \cos d(Tx, p) + \varphi'(C_x) \cos d(x, p)$$

and hence

$$\varphi'(C_x) \cos d(Tx, p) \leq \varphi'(C_x) \cos d(x, p).$$

Since $\varphi'(C_x) < 0$, it follows that

$$\cos d(Tx, p) \geq \cos d(x, p).$$

Furthermore, since $\cos t$ is decreasing for all $t \in [0, \pi/2[$, we get $d(Tx, p) \leq d(x, p)$. Thus we get the conclusion. ■

Theorem 4.2. *Let X be an admissible CAT(1) space, T a spherically vicinal mapping with φ from X into itself and $p \in X$. If a sequence $\{x_n\}$ of X satisfies $\mathcal{A}(\{x_n\}) = \{p\}$ and $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$, then p is a fixed point of T .*

Proof. The spherical vicinality of T implies that

$$\begin{aligned} &(\varphi'(C_{x_n}) + \varphi'(C_p)) \cos d(Tx_n, Tp) \\ &\leq \varphi'(C_p) \cos d(Tx_n, p) + \varphi'(C_{x_n}) \cos d(x_n, Tp) \end{aligned}$$

and hence

$$\begin{aligned} &\cos d(Tx_n, Tp) \\ &\geq \cos d(Tx_n, p) + \frac{\varphi'(C_{x_n})}{\varphi'(C_p)} (\cos d(x_n, Tp) - \cos d(Tx_n, Tp)). \end{aligned}$$

Using the nonexpansiveness of the cosine function, we get

$$\begin{aligned} &\cos d(Tx_n, Tp) \tag{4.1} \\ &\leq \cos d(Tx_n, p) + \frac{\varphi'(C_{x_n})}{\varphi'(C_p)} (\cos d(x_n, Tp) - \cos d(Tx_n, Tp)) \\ &\leq \cos d(Tx_n, p) + \frac{\varphi'(C_{x_n})}{\varphi'(C_p)} |d(x_n, Tp) - d(Tx_n, Tp)|. \end{aligned}$$

On the other hand, since $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$, we know that

$$\lim_{n \rightarrow \infty} (d(x_n, Tp) - d(Tx_n, Tp)) = 0.$$

Therefore, taking the lower limit of both sides of (4.1), we have

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \cos d(Tx_n, Tp) \\ &\leq \liminf_{n \rightarrow \infty} \left(\cos d(Tx_n, p) + \frac{\varphi'(C_{x_n})}{\varphi'(C_p)} |d(x_n, Tp) - d(Tx_n, Tp)| \right) \\ &= \liminf_{n \rightarrow \infty} \cos d(Tx_n, p) + \frac{\varphi'(1)}{\varphi'(C_p)} \times 0 \\ &= \liminf_{n \rightarrow \infty} \cos d(Tx_n, p). \end{aligned}$$

Since the cosine function is decreasing, we get

$$\cos \limsup_{n \rightarrow \infty} d(Tx_n, Tp) \geq \cos \limsup_{n \rightarrow \infty} d(Tx_n, p)$$

and hence

$$\limsup_{n \rightarrow \infty} d(Tx_n, Tp) \leq \limsup_{n \rightarrow \infty} d(Tx_n, p).$$

The assumption of $\{x_n\}$ implies that $\limsup_{n \rightarrow \infty} d(Tx_n, y) = \limsup_{n \rightarrow \infty} d(x_n, y)$ for all $y \in X$. Therefore it follows that

$$\limsup_{n \rightarrow \infty} d(x_n, Tp) \leq \limsup_{n \rightarrow \infty} d(x_n, p).$$

Since $\mathcal{A}(\{x_n\}) = \{p\}$, we get the conclusion. ■

Using Theorem 4.2, we can get the following result.

Corollary 4.3. *Every spherically vicinal mapping with φ on an admissible CAT(1) space is Δ -demiclosed.*

Proof. Let $\{x_n\}$ be a sequence of an admissible CAT(1) space satisfying $x_n \overset{\Delta}{\rightarrow} p$ and $d(Tx_n, x_n) \rightarrow 0$. By the definition of Δ -convergence, we know that $\mathcal{A}(\{x_n\}) = \{p\}$. Therefore, by Theorem 4.2, p is a fixed point of T . Thus we get the conclusion. ■

Lemma 4.4. *Let X be an admissible CAT(1) space and T a firmly spherically vicinal mapping with φ from X into itself. If $\mathcal{F}(T)$ is nonempty, then*

$$\cos d(Tx, x) \cos d(Tx, p) \geq \cos d(x, p)$$

for all $x \in X$ and $p \in \mathcal{F}(T)$.

Proof. Let $x \in X$ and $p \in \mathcal{F}(T)$. Since T is firmly spherically vicinal with φ , we have

$$\begin{aligned} &(\varphi'(C_x)C_x + \varphi'(1)) \cos d(Tx, p) \\ &\leq \varphi'(1) \cos d(Tx, p) + \varphi'(C_x) \cos d(x, p). \end{aligned}$$

Hence we obtain

$$\varphi'(C_x)C_x \cos d(Tx, p) \leq \varphi'(C_x) \cos d(x, p).$$

Dividing by $\varphi'(C_x)$, we get the conclusion. ■

Theorem 4.5. *Let X be an admissible CAT(1) space and T a firmly spherically vicinal mapping with φ from X into itself. If $\mathcal{F}(T)$ is nonempty, then T is asymptotically regular.*

Proof. Let $x \in X$ and $p \in \mathcal{F}(T)$. By Lemma 4.4, we know that

$$\cos d(T^{n+1}, T^n x) \geq \frac{\cos d(T^n x, p)}{\cos d(T^{n+1} x, p)}. \tag{4.2}$$

On the other hand, from Theorem 4.1, T is quasi-nonexpansive. Therefore we have

$$0 \leq d(T^n x, p) \leq d(T^{n-1} x, p) \leq \dots \leq d(x, p) < \frac{\pi}{2}.$$

Thus there exists $d \in [0, \pi/2[$ such that $\{d(T^n x, p)\}$ is convergent to d . Hence, taking the limit of both sides of (4.2), we get

$$1 \geq \lim_{n \rightarrow \infty} \cos d(T^{n+1} x, T^n x) \geq \lim_{n \rightarrow \infty} \frac{\cos d(T^n x, p)}{\cos d(T^{n+1} x, p)} = 1.$$

Therefore we have $\lim_{n \rightarrow \infty} d(T^{n+1} x, T^n x) = 0$. ■

Theorem 4.6. *Let X be an admissible complete CAT(1) space and T a firmly spherically vicinal mapping with φ from X into itself. If $\mathcal{F}(T)$ is nonempty, then $\{T^n x\}$ is Δ -convergent to an element of $\mathcal{F}(T)$ for each $x \in X$.*

Proof. Let $x \in X$. It follows from Theorem 4.1, we have

$$\inf_{y \in X} \limsup_{n \rightarrow \infty} d(T^n x, y) \leq \inf_{y \in \mathcal{F}(T)} \limsup_{n \rightarrow \infty} d(T^n x, y) \leq \inf_{y \in \mathcal{F}(T)} d(x, y) < \frac{\pi}{2}.$$

Therefore, by using Lemma 2.2, $\{T^n x\}$ has a Δ -convergent subsequence. Let $\{T^{n_i} x\}$ be a subsequence of $\{T^n x\}$ satisfying $T^{n_i} x \xrightarrow{\Delta} z$. Using Theorem 4.5, we get

$$\lim_{n \rightarrow \infty} d(T(T^n x), T^n x) = \lim_{n \rightarrow \infty} d(T^{n+1} x, T^n x) = 0.$$

Thus Corollary 4.3 implies that $z \in \mathcal{F}(T)$. It follows from Theorem 4.1 that

$$0 \leq d(T^n x, z) \leq d(T^{n-1} x, z) \leq \cdots \leq d(x, y) < \frac{\pi}{2}.$$

Therefore $\{d(T^n x, z)\}$ is convergent. Hence, by Lemma 2.3, $\{T^n x\}$ is Δ -convergent to an element of X . From Corollary 4.3 and Theorem 4.5, the Δ -limit of $\{T^n x\}$ belongs to $\mathcal{F}(T)$. ■

We remark that this result is related to the proximal point algorithm in admissible complete CAT(κ) spaces; for recent results, see [3, 6, 9–11].

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