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AN EXTRAGRADIENT ALGORITHM FOR STRONGLY PSEUDOMONOTONE EQUILIBRIUM PROBLEMS ON HADAMARD MANIFOLDS

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Abstract In this article, we introduce two extragradient algorithms for solving equilibrium problems in Hadamard manifolds. Furthermore, we prove that any sequence generated by the proposed algorithms converge to a solution of the equilibrium problems under suitable assumptions. A numerical experiment of the proposed algorithms are provided to support our convergence results.

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1. Introduction

The Equilibrium problem (EP) was early introduced by Fan [11] which is also known as the Fan's inequality and extensively developed by Blum and Oettli[3]. It plays a very important role in many fields such as variational inequalities, game theory, mathematical economics, optimization theory, and fixed point theory (see, for example [10, 20, 23, 27, 36] and the references). Let Ω be a nonempty closed convex subset of a real Hilbert space H. Let $S : \Omega \times \Omega \to \mathbb{R}$ be a bifunction, called *equilibrium bifunction* if and only if $S(x, x) = 0$, for all $x \in \Omega$. The equilibrium problem is to find $x^* \in \Omega$ such that

$$S(x^*, y) \geq 0, \quad \forall y \in \Omega. \tag{1.1}$$

A point x^* solving this problem is said to be an *equilibrium point* (called equilibria as well). There are many methods have been extensively studied for approximating solutions of the problem (1.1), for instance; the proximal point algorithm [26], the proximal-like algorithm (the extragradient algorithm) [15, 34], the subgradient algorithm [4, 14] and auxiliary problem principle [25].

One of the most popular of studied methods on approximate the equilibrium points is the proximal point method. The method was firstly introduced by Martinet [24] for convex minimization and further generalized by Rockafellar [30]. Later author studied on approximate zero point for a monotone operator in Hilbert spaces and shown the classical version. In 1999, Moudafi [26] extended the proximal point method to equilibrium problems for monotone bifunctions. Afterward, Konnov [19] proposed another kind of proximal point method with weaker conditions. The proximal point method is normally imposed to monotone equilibrium problems, i.e., the bifunction of an equilibrium problem has to be monotone. Then, each regularized subproblem becomes strongly monotone, and its solution exists and is unique. This property cannot be guaranteed if the equilibrium bifunction more generally monotone, for example, pseudomonotone.

On the other hand, another well known is the auxiliary problem principle is to develop a new problem which is equivalent and usually easier to work out than the original problem. This principle was initially introduced by Cohen [6] for optimization problems and then, utilized to variational inequality problems [7]. Mastroeni[25] further extended the auxiliary problem principle to equilibrium problems related to strongly monotone bifunctions.

Another one method based on the auxiliary problem principle was presented early by Flåm and Antipin [13] which was called proximal-like algorithm. The convergence results of this method was further extended and investigated the convergence of it under assumptions that the bifunctions are pseudomonotone and satisfy the Lipschitz-type condition [25], i.e., there exists two positive constants γ_1, γ_2 such that

$$S(x, y) + S(y, z) + \geq S(x, z) - \gamma_1 \|x - y\|^2 - \gamma_2 \|y - z\|^2,$$

for all $x, y, x \in \Omega$. The methods in [13, 25] are also called extragradient algorithms due to the work of Korpelevich [21]. The extragradient algorithm is described as follows: from an initial point $x^0 \in \Omega$, compute x^k, y^k, for each $k \in \mathbb{N}$ by

$$\begin{cases} y^k = \arg\min_{t \in \Omega} \{\lambda S(x^k, t) + G(x^k, t)\}, \\ x^{k+1} = \arg\min_{t \in \Omega} \{\lambda S(y^k, t) + G(x^k, t)\}, \end{cases}$$

where λ is a suitable fixed parameter and $G(x,y)$ is the Bregman distance function. The advantages of extragradient algorithm are that it can be applied to the class of psuedomontone bifunctions and two optimizations problems may be computed easily than the proximal point method.

Very recently, Hieu [16] has presented the extragradient algorithms for solving equilibrium problems where the bifunctions are strongly pseudomonotone and satisfy the Lipschitz-type condition which described as follows: from an initial point $x^0 \in \Omega$, compute x^k, y^k, for each $k \in \mathbb{N}$ by

$$\begin{cases} y^k = \underset{t\in\Omega}{\arg\min}\{\lambda_k S(x^k,t) + \frac{1}{2}\|x^k - t\|^2\}, \\ x^{k+1} = \underset{t\in\Omega}{\arg\min}\{\lambda_k S(t,y^k) + \frac{1}{2}\|x^k - t\|^2\}, \end{cases}$$

where $\{\lambda_k\}$ is is a non-increasing sequence satisfying the following conditions:

$$\text{(C1)} \lim_{k\to\infty} \lambda_k = 0, \quad \text{(C2)} \sum_{k=0}^{\infty} \lambda_k = +\infty.$$

The authors, deduce that any sequences generated by the proposed algorithm strongly converge to solutions of the equilibrium problems in Hilbert spaces without the prior knowledge of Lipschitz-type constants and any hybrid method. In [15], Hieu also considered the extragradient algorithms which a stepsize sequence is non-increasing, diminishing and non-summable for solving strongly pseudomnontone equilibrium problems in Hilbert spaces. Precisely, this algorithm is described as follows: from an initial point $x^0, y^0 \in \Omega$, compute x^k, y^k, for each $x \geq 1$ by

$$\begin{cases} x^{k+1} = \underset{t\in\Omega}{\arg\min}\{\lambda_k S(y^k,t) + \frac{1}{2}\|x^k - t\|^2\}, \\ y^{k+1} = \underset{t\in\Omega}{\arg\min}\{\lambda_{k+1} S(y^k,t) + \frac{1}{2}\|x^{k+1} - t\|^2\}, \end{cases}$$

where $\{\lambda_k\}$ is is a non-increasing sequence satisfying conditions (C1) and (C2).

During the last decade, there are many issues in nonlinear analysis such as fixed point theory, convex analysis, variational inequality, equilibrium theory, and optimization theory have been magnified from linear setting, namely, Banach space or Hilbert space, etc., to nonlinear system because the problems cannot be posted in the linear space and require a manifold structure (not necessary with linear structure). The main advantages of these extensions are that non-convex problems in the general sense transform into convex problems, and constrained optimization problems also transform into unconstrained optimization problems. Eigenvalue optimization problems [32] and geometric models for the human spine [1] are typical examples of the situation. Therefore, many authors have focused on extension and development of nonlinear problems techniques on the Riemannian manifold, see for examples [8, 12, 22, 33] and the reference therein.

In recent years, many researchers [5, 8, 18, 28, 29] extended the concepts and techniques of the equilibrium theory from linear spaces to Riemannian context. Especially, Colao et al. [8] was firstly introduced the equilibrium problems on Riemannian setting. Let M be an Hadamard manifold, C a nonempty closed geodesic convex subset of M, and $S : \Omega \times \Omega \to \mathbb{R}$ a bifunction satisfying $S(x,x) = 0$, for all $x \in \Omega$. Then the equilibrium problem on the Hadamard manifold is to find $x^* \in \Omega$ such that

$$S(x^*, y) \geq 0, \quad \forall y \in \Omega. \tag{EP}$$

We denote by $EP(S,\Omega)$ the set solution of problem (EP), we also suppose that the set $EP(S,\Omega)$ is nonempty. Moreover, they proved the existence of an equilibrium point for a biunction under suitable conditions and applied their results to solve variational inequality, fixed point and Nash equilibrium problems. Chaipunya and Kumam [5] considered the nonself of KKM lemma on Hadamard manifolds and applied their result to equilibrium problems. Jana and Nahak [18] extended concept of a mixed equilibrium problem to Hadamrad manifolfs and introduced an implicit and explicit method to solving the problem. Recently, Cruz Neto et al. [29] presented an extragradient algorithm to solving equilibrium problems on Hadamard manifolds. The extragradient algorithm is described as follows: from an initial point $x^0 \in \Omega$, compute x^k, y^k, for each $k \in \mathbb{N}$ by

$$\begin{cases} y^k = \arg\min\limits_{t\in\Omega}\{S(x^k,t) + \frac{1}{2\lambda_k}d^2(x^k,t)\}, \\ x^{k+1} = \arg\min\limits_{t\in\Omega}\{S(t,y^k) + \frac{1}{2\lambda_k}d^2(x^k,t)\}, \end{cases}$$

where $\{\lambda_k\} \subset (0,+\infty)$. The authors also proved that the proposed algorithm converges to a solution of equilibrium problems involving the bifunctions does not satisfy pseudomonotone.

Motivation mentioned above, and the works due to [15, 16, 29], the propose of the paper is to introduce two extragradient algorithms for solving the equilibrium problem (EP) regard to strongly pseudomonotone bifunctions and the Lipschitz-type condition on Hadamard manifolds. The convergence of the resulting algorithms is established under suitable conditions.

The paper is organized as follows : Section 2, we give some basic concept and fundamental results of Riemannian manifolds and some useful results for further use. Section 3, deals with proposing of two extragradient algorithms involving strongly pseudomonotone bifunctions and analysing the convergence results of the proposed algorithms on Hadamard manifolds. Section 4, we give numerical experiments to illustrate the computational performance on a test problem. In the last Section, contains conclusion of the main results.

2. Preliminaries

In this section, we recall some fundamental definitions, properties, useful results, and notations of Riemannian geometry. Readers refer to some textbooks [9, 31, 35] for more details.

Let M be a connected finite-dimensional manifold. For $p \in M$, we denote T_pM the *tangent space* of M at p which is a vector space of the same dimension as M, and by $TM = \bigcup_{p\in M} T_pM$ the *tangent bundle* of M. We always suppose that M can be endowed with a Riemannian metric $\langle\cdot,\cdot\rangle_p$, with corresponding norm denoted by $\|\cdot\|_p$, to become a *Riemannian manifold*. The angle $\angle_p(u,v)$ between $u,v \in T_pM$ $(u,v \neq 0)$ is set by $\cos\angle_p(u,v) = \dfrac{\langle u,v\rangle_p}{\|u\|\|v\|}$. If there is no confusion, we denote $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_p$, $\|\cdot\| := \|\cdot\|_p$ and $\angle(u,v) := \angle_p(u,v)$. Let $\gamma : [a,b] \to M$ be a piecewise smooth curve joining $\gamma(a) = p$ to

$\gamma(b) = q$, we define the length of the curve γ by using the metric as

$$L(\gamma) = \int_a^b \|\gamma^{'}(t)\|dt,$$

minimizing the length function over the set of all such curves, we obtain a Riemannian distance $d(p, q)$ which induces the original topology on M.

Let ∇ be a Levi-Civita connection associated to $(M, \langle\cdot,\cdot\rangle)$. Given γ a smooth curve, a smooth vector field X along γ is said to be *parallel* if $\nabla_{\gamma'} X = \mathbf{0}$. If $\gamma^{'}$ itself is parallel, we say that γ is a *geodesic*, and in this case $\|\gamma^{'}\|$ is a constant. When $\|\gamma^{'}\| = 1$, then γ is said to be *normalized.* A geodesic joining p to q in M is said to be a *minimal geodesic* if its length equals to $d(p, q)$.

A Riemannian manifold is complete if for any $p \in M$ all geodesic emanating from p are defined for all $t \in \mathbb{R}$. From the Hopf-Rinow theorem we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and every bounded closed subsets are compact.

Let M be a complete Riemannian manifold and $p \in M$. The exponential map $\exp_p : T_pM \to M$ is defined as $\exp_p v = \gamma_v(1, x)$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at p with velocity v (i.e., $\gamma_v(0, p) = p$ and $\gamma_v^{'}(0, p) = v$). Then, for any value of t, we have $\exp_p tv = \gamma_v(t, p)$ and $\exp_p \mathbf{0} = \gamma_v(0, p) = p$. Note that the exponential $\exp_p$ is differentiable on T_pM for all $p \in M$. It well known that the derivative $D\exp_p(\mathbf{0})$ of $\exp_p(\mathbf{0})$ is equal to the identity vector of T_pM. Therefore, by the inverse mapping theorem, there exists an inverse exponential map $\exp^{-1} : M \to T_pM$. Moreover, for any $p, q \in M$, we have $d(p, q) = \|\exp_p^{-1} q\|$.

A complete simply connected Riemannian manifold of non-positive sectional curvature is said to be an *Hadamard manifold.* Throughout the remainder of the paper, we always assume that M is a finite-dimensional Hadamard manifold. The following proposition is well-known and will be useful.

Proposition 2.1. [31] *Let $p \in M$. The $\exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $p, q \in M$ there exists a unique normalized geodesic joining p to q, which is can be expressed by the formula*

$$\gamma(t) = \exp_p t \exp_p^{-1} q, \quad \forall t \in [0, 1].$$

This proposition yields that M is diffeomorphic to the Euclidean space $\mathbb{R}^n$. Then, M has same topology and differential structure as $\mathbb{R}^n$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. One of the most important proprieties is illustrated in the following propositions.

A geodesic triangle $\triangle(p_1, p_2, p_3)$ of a Riemannian manifold M is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesic γ_i joining p_i to p_{i+1} where $i = 1, 2, 3$ (mod3).

Proposition 2.2. [31] *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in M. For each $i = 1, 2, 3$ (mod3), given $\gamma_i : [0, l_i] \to M$ the geodesic joining p_i to p_{i+1} and set $l_i := L(\gamma_i)$, $\alpha_i : \angle(\gamma_i^{'}(0), -\gamma_{i-1}^{'}(l_{i-1}))$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi; \tag{2.1}$$

$$l_i^2 + l_{i+1}^2 - 2l_il_{i+1}\cos\alpha_{i+1} \le l_{i-1}^2. \tag{2.2}$$

In the terms of the distance and the exponential map, the inequality (2.2) can be rewritten as

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2}\rangle \le d^2(p_{i-1}, p_i), \tag{2.3}$$

where $\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2}\rangle = d(p_i, p_{i+1})d(p_{i+1}, p_{i+2})\cos\alpha_{i+1}$.

Definition 2.3. A subset Ω is called *geodesic convex* if for every two points p and q in Ω, the geodesic joining p to q is contained in Ω, that is, if $\gamma : [a, b] \to M$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1-t)a + tb) \in \Omega$ for all $t \in [0, 1]$.

Definition 2.4. A real function f defined on M is called *geodesic convex* if for any geodesic γ of M, the composition function $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \le t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b),$$

where $a, b \in \mathbb{R}$, and $t \in [0, 1]$.

Definition 2.5. Let $f : M \to \mathbb{R}$ be a geodesic convex and $p \in M$. A vector $s \in T_pM$ is called a *subgradient* of f at p if and only if

$$f(q) \ge f(p) + \langle s, \exp_p^{-1} q\rangle, \quad \forall q \in M. \tag{2.4}$$

The set of all subgradients of f, denoted by $\partial f(p)$ is called the *subdifferential* of f at p, which is closed geodesic convex (possibly empty) set. Let $D(\partial f)$ denote the domain of ∂f defined by $D(\partial f) = \{p \in M \mid \partial f(p) \ne \emptyset\}$. The following proposition is guaranteed the existence of subgradients for geodesic convex functions.

Proposition 2.6. [12] *Let M be a Hadamard manifolds and $f : [a, b] \to \mathbb{R}$ be a geodesic convex. Then, for all $p \in M$, the subdifferential $\partial f(p)$ of f at p is nonempty. That is, $D(\partial f) = M$.*

Let $f : M \to \mathbb{R}$ be a geodesic convex, proper, and lower semicontinuous. The *proximal point algorithm* generates, for a initial point $p^0 \in M$, a sequence $\{p^k\} \subset M$ is defined by the following:

$$p^{k+1} = \arg\min_{t \in M} \left\{ f(t) + \frac{\lambda_k}{2} d^2(p^k, t) \right\}, \tag{2.5}$$

where $\{\lambda_k\} \subset (0, +\infty)$.

Theorem 2.7. [12] *Let $f : M \to \mathbb{R}$ be a geodesic convex, proper, and lower semicontinuous. Then the sequence $\{p^k\}$ generated by (2.5) is well defined, and characterized by*

$$\lambda_k(\exp_{p^{k+1}}^{-1} p^k) \in \partial f(p^{k+1}).$$

The distance function of a point $p \in M$ to a nonempty, closed and geodesic convex set $\Omega \subset M$ is define by

$$d_\Omega(p) := \inf\{d(p, q) : \forall q \in \Omega\}.$$

Remark 2.8. [31] It is important to mention that for all $q \in M, p \mapsto d(p, q)$ is continuous and geodesic convex function.

Next, we recall some concepts of monotonicity of a bifunction (see [3, 27] for further details).

Definition 2.9. A bifunction $S : \Omega \times \Omega \to \mathbb{R}$ is said to be

(1) *monotone* if and only if $S(p,q) + S(q,p) \leq 0$ for all $(p,q) \in \Omega \times \Omega$;

(2) *pseudomonotone* if and only if for all $(p,q) \in \Omega \times \Omega$,
$S(p,q) \geq 0 \Longrightarrow S(q,p) \leq 0$;

(3) *strongly monotone* if and only if there exists a positive constant ρ such that
$S(p,q) + S(q,p) \leq -\rho d^2(p,q)$ for all $(p,q) \in \Omega \times \Omega$;

(4) *strongly pseudomonotone* if and only if there exists a positive constant ρ such that
$S(p,q) \geq 0 \Longrightarrow S(q,p) \leq -\rho d^2(p,q)$ for all $(p,q) \in \Omega \times \Omega$.

Remark 2.10. From Definition 2.9, it easy to see that the following implications hold the following:

$$(3) \Longrightarrow (1) \Longrightarrow (2) \text{ and } (1) \Longrightarrow (4) \Longrightarrow (3)$$

The converse does not hold even in a linear context.

A Lipschitz-type condition is often use in construct the convergence of extragradient methods for equilibrium problems which is introduced by Mastroeni [25].

Definition 2.11. [25] A bifunction $S : \Omega \times \Omega \to \mathbb{R}$ is said to be *Lipschitz-type condition* on Ω if there exist two positive constants γ_1 and γ_2 such that

$$S(p,q) + S(q,r) \geq S(p,r) - \gamma_1 d^2(p,q) - \gamma_2 d^2(q,r),$$

for all $p,q,r \in \Omega$.

3. Extragradient Algorithm for Equilibrium Problem

In this section, we present two extragradient algorithms involving strongly pseudomonotone for equilibrium problems (EP) on Hadamard manifolds.

From now on $\Omega \subset M$ denote a nonempty closed geodesic convex set, unless explicitly stated otherwise. Let $S : \Omega \times \Omega \to \mathbb{R}$ be a bifunction satisfying $S(x,x) = 0$, for all $x \in \Omega$. For solving the problem (EP), we consider the following hypothesises regrading the bifunction;

(H1) For all $x \in \Omega$, $S(x,x) \geq 0$;

(H2) For every $x \in \Omega$, $y \mapsto S(x,y)$ are geodesic convex and lower semicontinuous;

(H3) S satisfies the Lipschitz-type condition;

(H4) S is strongly pseudomonotone.

The following algorithm is the first extragradient algorithm for finding the solution of the problem (EP).

The following remark gives us a stopping criterion of Algorithm 1.

Algorithm 1

Initialization: Choose $x^0, y^0 \in \Omega$ and nonincreasing sequence $\{\lambda_k\} \subset (0, +\infty)$ satisfying the following hypotheses:

$$(\text{A1}): \lim_{k\to\infty} \lambda_k = 0, \quad (\text{A2}): \sum_{k=0}^{\infty} \lambda_k = +\infty.$$

Iterative Steps: Given $x^k, y^k \in \Omega$, calculate x^{k+1} and y^{k+1} as follows:

Step 1. Compute

$$x^{k+1} = \arg\min_{t\in\Omega} \left\{ S(y^k, t) + \frac{1}{2\lambda_k} d^2(x^k, t) \right\}.$$

If $x^{k+1} = y^k = x^k$ then stop and x^k is the solution of problem (EP). Otherwise,

Step 2. Compute

$$y^{k+1} = \arg\min_{t\in\Omega} \left\{ S(y^k, t) + \frac{1}{2\lambda_{k+1}} d^2(x^{k+1}, t) \right\}.$$

Set $k =: k + 1$ and go back to **Step 1.**

Remark 3.1. Under hypotheses (H1), (H2) and if $x^{k+1} = y^k = x^k$ then x^k is a solution of (EP) on Ω. From definition of y^{k+1}, $y \mapsto S(x, y)$ is geodesic convex and Theorem 2.7, we obtain

$$\begin{aligned} S(y^k, y) &\geq S(y^k, x^{k+1}) + \frac{1}{\lambda_k} \langle \exp^{-1}_{x^{k+1}} x^k, \exp^{-1}_{x^{k+1}} y \rangle \\ &\geq 0, \quad \forall y \in \Omega. \end{aligned}$$

Hence, $x^k \in EP(S, \Omega)$. A similar stopping criterion of Algorithm 1 is that if $y^{k+1} = y^k = x^{k+1}$, then x^{k+1} is a solution of the problem (EP).

Proposition 3.2. *The sequence generated by the Algorithm 1 is well defined.*

Thank to Remark 3.1, if Algorithm 1 terminates then we can found a solution of (EP) on Ω. On the other hand, if Algorithm 1 does not stop, we have the following results.

Lemma 3.3. *From Algorithm 1 we have the following use inequality.*

$$S(y^k, y) \geq S(y^k, x^{k+1}) + \frac{1}{2\lambda_k} [d^2(x^k, x^{k+1}) + d^2(x^{k+1}, y) - d^2(x^k, y)], \quad \forall y \in \Omega. \tag{3.1}$$

Proof. From definition of x^{k+1}, $y \mapsto S(x, y)$ is geodesic convex and Theorem 2.7, we get

$$S(y^k, y) \geq S(y^k, x^{k+1}) + \frac{1}{\lambda_k} \langle \exp^{-1}_{x^{k+1}} x^k, \exp^{-1}_{x^{k+1}} y \rangle, \quad \forall y \in \Omega. \tag{3.2}$$

Let $\triangle(x^k, x^{k+1}, y) \subseteq M$ be the geodesic triangle and using (2.3), we obtain

$$2\langle \exp^{-1}_{x^{k+1}} x^k, \exp^{-1}_{x^{k+1}} y \rangle \geq d^2(x^k, x^{k+1}) + d^2(x^{k+1}, y) - d^2(x^k, y). \tag{3.3}$$

Combining (3.2) into (3.3), we have

$$S(y^k, y) \geq S(y^k, x^{k+1}) + \frac{1}{2\lambda_k} [d^2(x^k, x^{k+1}) + d^2(x^{k+1}, y) - d^2(x^k, y)], \quad \forall y \in \Omega.$$

■

Lemma 3.4. *From Algorithm 1 we have the following use inequality.*

$$S(y^{k-1}, y) \geq S(y^{k-1}, y^k) + \frac{1}{2\lambda_k}[d^2(x^k, y^k) + d^2(y^k, y) - d^2(x^k, y)], \quad \forall y \in \Omega. \tag{3.4}$$

Proof. From definition of y^k, $y \mapsto S(x, y)$ is geodesic convex and Theorem 2.7, we get

$$S(y^{k-1}, y) \geq S(y^{k-1}, y^k) + \frac{1}{\lambda_k}\langle \exp^{-1}_{y^k} x^k, \exp^{-1}_{y^k} y\rangle, \quad \forall y \in \Omega. \tag{3.5}$$

Let $\triangle(x^k, y^k, y) \subseteq M$ be the geodesic triangle and using (2.3), we obtain

$$2\langle \exp^{-1}_{y^k} x^k, \exp^{-1}_{y^k} y\rangle \geq d^2(x^k, y^k) + d^2(y^k, y) - d^2(x^k, y). \tag{3.6}$$

Combining (3.5) into (3.6), we have

$$S(y^{k-1}, y) \geq S(y^{k-1}, y^k) + \frac{1}{2\lambda_k}[d^2(x^k, y^k) + d^2(y^k, y) - d^2(x^k, y)], \quad \forall y \in \Omega.$$

■

Lemma 3.5. *Suppose that the hypotheses (H1)–(H4). Then, for all $p \in EP(S, \Omega)$, we have*

$$\begin{aligned} d^2(x^{k+1}, p) \quad \leq \quad & d^2(x^k, p) + 4\lambda_k\gamma_1 d^2(y^{k-1}, x^k) - (1 - 4\lambda_k\gamma_1)d^2(x^k, y^k) \\ & -(1 - 2\lambda_k\gamma_2)d^2(y^k, x^{k+1}) - 2\rho\lambda_k d^2(y^k, p) \end{aligned} \tag{3.7}$$

Proof. It follows from the Lemma 3.3, and letting $y = p \in \Omega$ into (3.1), we obtain

$$S(y^k, p) \geq S(y^k, x^{k+1}) + \frac{1}{2\lambda_k}[d^2(x^k, x^{k+1}) + d^2(x^{k+1}, p) - d^2(x^k, p)]. \tag{3.8}$$

Further, letting $y = x^{k+1} \in \Omega$ into (3.4), in Lemma 3.4, we obtain

$$S(y^{k-1}, x^{k+1}) \geq S(y^{k-1}, y^k) + \frac{1}{2\lambda_k}[d^2(x^k, y^k) + d^2(y^k, x^{k+1}) - d^2(x^k, x^{k+1})]. \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\begin{aligned} S(y^k, p) + S(y^{k-1}, x^{k+1}) \quad \geq \quad & S(y^k, x^{k+1}) + S(y^{k-1}, y^k) \\ & + \frac{1}{2\lambda_k}[d^2(x^{k+1}, p) - d^2(x^k, p) + d^2(x^k, y^k) + d^2(y^k, x^{k+1})]. \end{aligned}$$

Following above inequality, we obtain

$$\begin{aligned} d^2(x^{k+1}, p) \quad \leq \quad & 2\lambda_k[S(y^k, p) + S(y^{k-1}, x^{k+1}) - S(y^k, x^{k+1}) - S(y^{k-1}, y^k)] \\ & + d^2(x^k, p) - d^2(x^k, y^k) - d^2(y^k, x^{k+1}). \end{aligned} \tag{3.10}$$

Since S satisfies Lipschitz-type condition, we get

$$S(y^k, x^{k+1}) + S(y^{k-1}, y^k) \geq S(y^{k-1}, x^{k+1}) - \gamma_1 d^2(y^{k-1}, y^k) - \gamma_2 d^2(y^k, x^{k+1}). \tag{3.11}$$

Recall $p \in EP(S, \Omega)$ then $S(p, y^k) \geq 0$. From S is strongly pseudomonotone, then there exists $\rho > 0$ such that

$$S(y^k, p) \leq -\rho d^2(y^k, p). \tag{3.12}$$

Substitution (3.11) and (3.12) into (3.10), we get

$$\begin{aligned} d^2(x^{k+1},p) &\leq 2\lambda_k[-\rho d^2(y^k,p)+\gamma_1 d^2(y^{k-1},y^k)+\gamma_2 d^2(y^k,x^{k+1})] \\ &\quad +d^2(x^k,p)-d^2(x^k,y^k)-d^2(y^k,x^{k+1}) \\ &\leq 2\lambda_k[-\rho d^2(y^k,p)+2\gamma_1 d^2(y^{k-1},x^k)+2\gamma_1 d^2(x^k,y^k)+\gamma_2 d^2(y^k,x^{k+1})] \\ &\quad +d^2(x^k,p)-d^2(x^k,y^k)-d^2(y^k,x^{k+1}) \\ &= d^2(x^k,p)+4\lambda_k\gamma_1 d^2(y^{k-1},x^k)-(1-4\lambda_k\gamma_1)d^2(x^k,y^k) \\ &\quad -(1-2\lambda_k\gamma_2)d^2(y^k,x^{k+1})-2\lambda_k\rho d^2(y^k,p). \end{aligned}$$

■

Theorem 3.6. *Let $p \in EP(S,\Omega)$ and $S:\Omega\times\Omega\to\mathbb{R}$ be a bifunction satisfying all hypotheses (H1)–(H4). Then, the sequences $\{x^k\}$ and $\{y^k\}$ generated by Algorithm 1 converge to p.*

Proof. From Lemma 3.5, adding $4\lambda_{k+1}\gamma_1 d^2(y^k,x^{k+1})$ into (3.7), we have

$$\begin{aligned} d^2(x^{k+1},p)+4\lambda_{k+1}\gamma_1 d^2(y^k,x^{k+1}) &\leq d^2(x^k,p)+4\lambda_k\gamma_1 d^2(y^{k-1},x^k) \\ &\quad -(1-4\lambda_k\gamma_1)d^2(x^k,y^k) \\ &\quad -(1-4\lambda_{k+1}\gamma_1-2\lambda_k\gamma_2)d^2(y^k,x^{k+1}) \\ &\quad -2\rho\lambda_k d^2(y^k,p), \end{aligned}$$

which implies that

$$a_{k+1} \leq a_k - b_k - \lambda_k c_k, \tag{3.13}$$

where

$$\begin{aligned} a_k &= d^2(x^k,p)+4\lambda_k\gamma_1 d^2(y^{k-1},x^k) \\ b_k &= (1-4\lambda_k\gamma_1)d^2(x^k,y^k)+(1-4\lambda_{k+1}\gamma_1-2\lambda_k\gamma_2)d^2(y^k,x^{k+1}) \\ c_k &= 2\rho d^2(y^k,p). \end{aligned}$$

We have $a_k \geq 0$ and $c_k \geq 0$ for all $k \geq 0$. Given ξ be fixed in $(0,1)$. Since $\lim_{k\to\infty}\lambda_k = 0$, there exists $k_0 \geq 0$ such that for all $k \geq k_0$,

$$0<\xi\leq 1-4\lambda_k\gamma_1<1 \text{ and } 0<\xi\leq 1-4\lambda_{k+1}\gamma_1-2\lambda_k\gamma_2<1, \tag{3.14}$$

which, together with the definition of b_k, yields that $b_k \geq 0$ for all $k \geq k_0$. Thus, from (3.13), we get

$$0 \leq a_{k+1} \leq a_k,$$

for all $k \geq k_0$. This implies that $\lim_{k\to\infty} a_k$ exists in $\mathbb{R}$. Hence, from the definition of a_k, the sequence $\{d^2(x^k,p)\}$ is bounded, and thus, $\{x^k\}$ is also bounded. Moreover, from (3.13), we obtain

$$b_k + \lambda_k c_k \leq a_k - a_{k+1}, \tag{3.15}$$

for all $k \geq k_0$. Let $K > k_0$ be fixed. Summing up (3.15) for $k = k_0, \ldots, K$, we get

$$\sum_{k=k_0}^{K} b_k + \sum_{k=k_0}^{K} \lambda_k c_k \leq a_{k_0} - a_{K+1} \leq a_{k_0}.$$

Thus, letting $K \to \infty$, we obtain $\sum_{k=k_0}^{\infty} b_k + \sum_{k=k_0}^{\infty} \lambda_k c_k < +\infty$. This implies that

$$(\mathrm{S1}) : \sum_{k=k_0}^{\infty} b_k < +\infty, \text{ and } (\mathrm{S2}) : \sum_{k=k_0}^{\infty} \lambda_k c_k < +\infty.$$

From the definition of b_k, (3.14) and (S1), we deduce that

$$\lim_{k\to\infty} d^2(x^k, y^k) = \lim_{k\to\infty} d^2(y^k, x^{k+1}) = 0. \tag{3.16}$$

From the boundedness of $\{x^k\}$, implies that $\{y^k\}$ is bounded. Hence, from the definition of a_k, $\lim_{k\to\infty} \lambda_k = 0$ and $\lim_{k\to\infty} a_k \in \mathbb{R}$, we have

$$\lim_{k\to\infty} d^2(x^k, p) \in \mathbb{R}. \tag{3.17}$$

From (S2) and $\sum_{k=k_0}^{\infty} \lambda_k = +\infty$, we obtain $\liminf_{k\to\infty} c_k = 0$. Thus from definition of c_k and $\rho > 0$, we also have that $\liminf_{k\to\infty} d^2(y^k, p) = 0$. Using (3.16) implies that $\liminf_{k\to\infty} d^2(x^k, p) = 0$. From (3.17), we get $\lim_{k\to\infty} d^2(x^k, p) = 0$. Furthermore, the sequence $\{y^k\}$ also converges to p via (3.16). Therefore, the proof is completed. ■

The following algorithm is the second extragradient algorithm for finding the solution of the problem (EP).

Algorithm 2

Initialization: Choose $x^0 \in \Omega$ and nonincreasing sequence $\{\lambda_k\} \subset (0, +\infty)$ satisfying the following hypothesis

$$(\mathrm{A1}) : \lim_{k\to\infty} \lambda_k = 0, \quad (\mathrm{A2}) : \sum_{k=0}^{\infty} \lambda_k = +\infty.$$

Iterative Steps: Given $x^k \in \Omega$, calculate y^k and x^{k+1} as follows:

Step 1. Compute

$$y^k = \arg\min_{t\in\Omega} \left\{ S(x^k, t) + \frac{1}{2\lambda_k} d^2(x^k, t) \right\}.$$

If $y^k = x^k$ then stop and x^k is the solution of problem (EP). Otherwise,

Step 2. Compute

$$x^{k+1} = \arg\min_{t\in\Omega} \left\{ S(y^k, t) + \frac{1}{2\lambda_k} d^2(x^k, t) \right\}.$$

Set $k =: k + 1$ and go back to **Step 1.**

The following remark gives us a stopping criterion of Algorithm 2.

Remark 3.7. Under hypotheses (H1), (H2) and if $y^k = x^k$ then x^k is a solution of EP on Ω. From definition of y^k, $y \mapsto S(x, y)$ is geodesic convex and Theorem 2.7, we obtain

$$\begin{aligned} S(x^k, y) &\geq S(x^k, y^k) + \frac{1}{\lambda_k} \langle \exp_{y^k}^{-1} x^k, \exp_{y^k}^{-1} y \rangle \\ &\geq 0, \quad y \in \Omega. \end{aligned}$$

Hence, $x^k \in EP(S, \Omega)$.

Proposition 3.8. *The sequence generated by the Algorithm 2 is well defined.*

Thank to Remark 3.7, if Algorithm 2 terminates then we can found a solution of (EP) on Ω. On the other hand, if Algorithm 2 does not stop, we have the following results.

Lemma 3.9. *From Algorithm 2 we have the following use inequality.*

$$S(y^k, y) \geq S(y^k, x^{k+1}) + \frac{1}{2\lambda_k}[d^2(x^k, x^{k+1}) + d^2(x^{k+1}, y) - d^2(x^k, y)], \quad \forall y \in \Omega. \tag{3.18}$$

Proof. Follow the same step as in the proof of Lemma 3.3 ■

Lemma 3.10. *From Algorithm 2 we have the following use inequality.*

$$S(x^k, y) \geq S(x^k, y^k) + \frac{1}{2\lambda_k}[d^2(x^k, y^k) + d^2(y^k, y) - d^2(x^k, y)], \quad \forall y \in \Omega. \tag{3.19}$$

Proof. Follow the same step as in the proof of Lemma 3.3 ■

Lemma 3.11. *Suppose that the hypotheses (H1)–(H4). Then, for all $p \in EP(S, \Omega)$, we have*

$$\begin{aligned} d^2(x^{k+1}, p) \quad \leq \quad & d^2(x^k, p) - (1 - 2\lambda_k\gamma_1)d^2(x^k, y^k) - (1 - 2\lambda_k\gamma_2)d^2(x^{k+1}, y^k) \\ & -2\rho\lambda_k d^2(y^k, p) \end{aligned} \tag{3.20}$$

Proof. It follows from the Lemma 3.9, and letting $y = p \in \Omega$ into (3.18), we obtain

$$S(y^k, p) \geq S(y^k, x^{k+1}) + \frac{1}{2\lambda_k}[d^2(x^k, x^{k+1}) + d^2(x^{k+1}, p) - d^2(x^k, p)]. \tag{3.21}$$

Further, letting $y = x^{k+1} \in \Omega$ into (3.19), in Lemma 3.10, we obtain

$$S(x^k, x^{k+1}) \geq S(x^k, y^k) + \frac{1}{2\lambda_k}[d^2(x^k, y^k) + d^2(y^k, x^{k+1}) - d^2(x^k, x^{k+1})]. \tag{3.22}$$

Combining (3.21) and (3.22), we get

$$\begin{aligned} S(y^k, p) + S(x^k, x^{k+1}) \quad \geq \quad & S(y^k, x^{k+1}) + S(x^k, y^k) \\ & + \frac{1}{2\lambda_k}[d^2(x^{k+1}, p) - d^2(x^k, p) + d^2(x^k, y^k) + d^2(y^k, x^{k+1})]. \end{aligned}$$

It follows from above inequality, we obtain

$$\begin{aligned} d^2(x^{k+1}, p) \quad \leq \quad & 2\lambda_k[S(y^k, p) + S(x^k, x^{k+1}) - S(y^k, x^{k+1}) - S(x^k, y^k)] \\ & + d^2(x^k, p) - d^2(x^k, y^k) - d^2(y^k, x^{k+1}). \end{aligned} \tag{3.23}$$

Since S satisfies Lipschitz-type condition, we get

$$S(x^k, y^k) + S(y^k, x^{k+1}) \geq S(x^k, x^{k+1}) - \gamma_1 d^2(x^k, y^k) - \gamma_2 d^2(y^k, x^{k+1}). \tag{3.24}$$

Recall $p \in EP(S, \Omega)$ then $S(p, y^k) \geq 0$. From S is strongly pseudomonotone, then there exists $\rho > 0$ such that

$$S(y^k, p) \leq -\rho d^2(y^k, p). \tag{3.25}$$

Taking (3.24) and (3.25) into (3.23), we get

$$\begin{aligned} d^2(x^{k+1},p) &\leq 2\lambda_k[-\rho d^2(y^k,p)+\gamma_1 d^2(x^k,y^k)+\gamma_2 d^2(y^k,x^{k+1})]+d^2(x^k,p)-d^2(x^k,y^k) \\ &\quad -d^2(y^k,x^{k+1}) \\ &= d^2(x^k,p)-(1-2\lambda_k\gamma_1)d^2(x^k,y^k)-(1-2\lambda_k\gamma_2)d^2(x^{k+1},y^k) \\ &\quad -2\lambda_k\rho d^2(y^k,p). \end{aligned}$$

■

Theorem 3.12. *Let $p \in EP(S,\Omega)$ and $S : \Omega \times \Omega \to \mathbb{R}$ be a bifunction satisfying all hypotheses (H1)–(H4). Then, the sequence $\{x^k\}$ generated by Algorithm 2 converges to p.*

Proof. From Lemma 3.11, we have

$$\begin{aligned} d^2(x^{k+1},p) &\leq d^2(x^k,p)-(1-2\lambda_k\gamma_1)d^2(x^k,y^k)-(1-2\lambda_k\gamma_2)d^2(x^{k+1},y^k) \\ &\quad -2\rho\lambda_k d^2(y^k,p). \end{aligned} \tag{3.26}$$

Since $\lim_{k\to\infty}\lambda_k = 0$, there exists k_0 such that $1-2\lambda_k\gamma_2 \geq 0$, $1-2\lambda_k\gamma_1 \geq 2\rho\lambda_k \geq 0$ and $\rho\lambda_k < 1$ for all $k \geq k_0$. These together with (3.26) imply that, for all $k \geq k_0$, we obtain

$$\begin{aligned} d^2(x^{k+1},p) &\leq d^2(x^k,p)-(1-2\lambda_k\gamma_1)d^2(x^k,y^k)-2\rho\lambda_k d^2(y^k,p) \\ &\leq d^2(x^k,p)-2\rho\lambda_k d^2(x^k,y^k)-2\rho\lambda_k d^2(y^k,p) \\ &= d^2(x^k,p)-2\rho\lambda_k[d^2(x^k,y^k)+d^2(y^k,p)] \\ &\leq d^2(x^k,p)-\rho\lambda_k[d(x^k,y^k)+d(y^k,p)]^2 \quad [2(a^2+b^2)\geq(a+b)^2, \forall a,b\in\mathbb{R}] \\ &\leq d^2(x^k,p)-\rho\lambda_k d^2(x^k,p), \end{aligned}$$

the last inequality is true from the triangle inequality. So, we have

$$d^2(x^{k+1},p) \leq d^2(x^k,p)-\rho\lambda_k d^2(x^k,p), \quad \forall k \geq k_0. \tag{3.27}$$

Furthermore, we fixed $K \geq k_0$ and consider (3.27) for $k = k_0,\ldots,K$. Summing up them, we obtain

$$d^2(x^{K+1},p) \leq d^2(x^{k_0},p)-\rho\sum_{k=k_0}^{K}\lambda_k d^2(x^k,p).$$

Hence,

$$\rho\sum_{k=k_0}^{K}\lambda_k d^2(x^k,p) \leq d^2(x^{k_0},p)-d^2(x^{K+1},p). \tag{3.28}$$

From relation (3.27), we also have $d^2(x^{k+1},p) \leq d^2(x^k,p)$ for all $k = k_0,\ldots,K$. Thus,

$$d^2(x^K,p) \leq d^2(x^k,p), \quad \forall k = k_0,\ldots,K, \tag{3.29}$$

Form (3.28) and (3.29), imply that

$$\begin{aligned} \rho\left(\sum_{k=k_0}^{K}\lambda_k\right)d^2(x^K,p) &= \rho\sum_{k=k_0}^{K}(\lambda_k d^2(x^K,p)) \\ &\leq \rho\sum_{k=k_0}^{K}(\lambda_k d^2(x^k,p)) \\ &\leq d^2(x^{k_0},p)-d^2(x^{K+1},p). \end{aligned}$$

This is true for all $K \geq k_0$. From condition (A2), $\sum_{k=0}^{\infty} \lambda_k = +\infty$, it follows from the last inequality that

$$\lim_{K\to\infty} d^2(x^K, p) = 0.$$

Hence, the sequence $\{x^k\}$ converges to p in $EP(S, \Omega)$. Therefore, the proof is completed. ■

4. Computational experiments

In this section, we check out the performance of the proposed algorithms, we show some numerical experiments involving the equilibrium problems (EP) relative to strongly pseudomonotone bifunction.

Following form [2, Example 1], let $\mathbb{R}^{++} = \{x \in \mathbb{R} : x > 0\}$ and $M = (\mathbb{R}^{++}, \langle\cdot,\cdot\rangle)$ be the Riemanian manifold with the metric $\langle x, y\rangle := xy$. Then the sectional curvature of M is 0, and the tangent space at $x \in M$, denoted by T_xM, equals $\mathbb{R}$. The Riemannian distance $d : M \times M \to \mathbb{R}^+$ is defined by

$$d(x,y) := \left|\ln\left(\frac{x}{y}\right)\right|.$$

Thus M is a Hadamard manifold. Let $\gamma : [0, 1] \to M$ be a geodesic starting from $x = \gamma(0)$ with velocity $v = \gamma^{'}(0) \in T_xM$ defined by

$$\gamma(t) := xe^{\frac{vt}{x}}.$$

Then

$$\exp_x tv = xe^{\frac{vt}{x}}.$$

For all $x, y \in M$, we have

$$y = \exp_x\left(d(x,y)\frac{\exp_x^{-1} y}{d(x,y)}\right) = xe^{\left(\frac{\exp_x^{-1} y}{xd(x,y)}\right)d(x,y)} = xe^{\frac{\exp_x^{-1} y}{x}},$$

and therefore, the inverse of exponential map is

$$\exp_x^{-1} y = x\ln\left(\frac{y}{x}\right).$$

For further details, see [9, 35].

Next, we consider an extension of a Nash-Cournot oligopolistic equilibrium model [10]. We suppose that there are n companies. Let x be a vector whose entry x_i stands for the quantity of the commodity produced by company i. Next, we assume that the price $p_i(s)$ is a decreasing affine function of $s = \sum_{i=1}^{n} x_i$ such as $p_i(s) = \alpha_i - \beta_i s$, where $\alpha_i, \beta_i \geq 0$. Then the profit made by company i is given by $S_i(x) = p_i(s)x_i - c_i(x_i)$, where $c_i(x_i)$ is the tax and fee for generating x_i. Assume that $\Omega_i = [x_{i,\min}, x_{i,\max}]$ is the strategy set of company i. Thus the strategy set of the model is $\Omega = \Omega_1 \times \cdots \times \Omega_n$. Actually, each company seeks to maximize its profit. A commonly used approach to this model is based upon the famous Nash equilibrium concept.

We recall that a point $x^* \in \Omega = \Omega_1 \times \cdots \times \Omega_n$ is an equilibrium point of the model if

$$S_i(x^*) \geq S_i(x^*[x_i]),\ \forall x_i \in \Omega_i,\ \forall i = 1, \ldots, n,$$

where the vector $x^*[x_i]$ stands for the vector obtained from x^* by replacing x_i^* with x_i. Given $S(x, y) = \phi(x, y) - \phi(x, x)$, where $\phi(x, y) = -\sum_{i=1}^{n} S_i(x[y_i])$ the problem of finding a Nash equilibrium point of the model can be formulated as:

$$\text{Find } x^* \in \Omega \text{ such that } S(x^*, x) \geq 0,\ \forall x \in \Omega.$$

Now, suppose that the tax-fee function $c_i(x_i)$ is increasing and affine for every i. This assumption means that both of tax and fee for producing a unit are increasing as the quantity of the production gets larger. Herein, the bifunction S can be in the form

$$S(x, y) = \langle Ax + By + q, y - x\rangle,$$

where $q \in \mathbb{R}^n$ and A, B are two matrices of order n such that B is symmetric positive semi-definite and $B - A$ is symmetric negative semi-definite. We consider here that $B - A$ is symmetric negative definite. From the property of $B - A$, if $S(x, y) \geq 0$, we have

$$\begin{aligned} S(y, x) &\leq S(y, x) + S(x, y) \\ &= \langle Ay + Bx + q, x - y\rangle + \langle Ax + By + q, y - x\rangle \\ &= \langle (A - B)y + (B - A)x, x - y\rangle \\ &= (x - y)^T (B - A)(x - y) \\ &\leq -\rho d^2(x, y), \end{aligned}$$

where $\rho > 0$. Thus S is strongly pseudomonotone or assumption (H4) holds for S. Moreover, for verifying S satisfies the Lipschitz-type condition, see, e.g., [17, Lemma 6], so (H3) is verified. Assumption (H1) and (H2) are automatically fulfilled and so Algorithm 1 and Algorithm 2 can be applied in this case.

For numerical experiment: we consider four companies, defined by

- First company, price $p_1(s) = 100 - 0.1s$, tax $c_1(x_1) = 20x_1$ and strategy set of company $\Omega_1 = [1000, 2000]$.
- Second company, price $p_2(s) = 110 - 0.2s$, tax $c_2(x_2) = 15x_2 + 100$ and strategy set of company $\Omega_2 = [500, 2500]$.
- Third company, price $p_3(s) = 100 - 0.15s$, tax $c_3(x_3) = 17x_3$ and strategy set of company $\Omega_3 = [800, 1500]$.
- Forth company, price $p_4(s) = 115 - 0.05s$, tax $c_4(x_4) = 20x_4 + 75$ and strategy set of company $\Omega_4 = [500, 3000]$.

We random x^1 and y^1. The starting points are $x^0 = y^0 = (1000, 1000, 1000, 1000)^T \in \mathbb{R}^4$. All the program are written in Matlab R2016b and computed on PC Intel(R) Core(TM) i7 @1.80 GHz, Ram 8.00 GB.

Next, we will study the numerical behavior of Algorithm 1 and Algorithm 2. Four groups of sequences λ_k used in the experiments are:

(I) $\lambda_k = \frac{1}{(k+1)^p}$, $p \in \{0.3, 0.5, 0.8, 1\}$;

(II) $\lambda_k = \frac{1}{\log^p(k+3)}$, $p \in \{0.6, 1, 3, 5\}$;

(III) $\lambda_k = \frac{1}{(k+1)\log^p(k+3)}$, $p \in \{1, 3, 5, 7\}$;

(IV) $\lambda_k = \frac{\log^p(k+3)}{(k+1)}$, $p \in \{1, 3, 5, 7\}$;

It is emphasized that all these sequences of λ_k are to satisfy conditions (A1) and (A2).

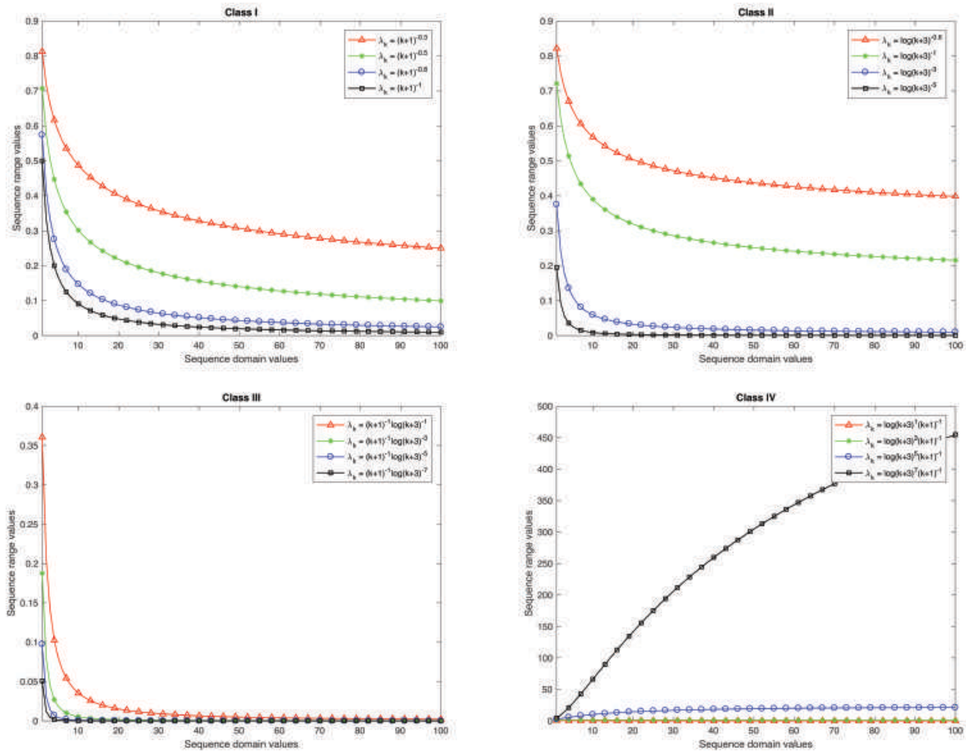

FIGURE 1. The behaviour of step size sequences

From Algorithm 1, we see that if $x^{k+1} = y^k = x^k$, then x^k is the solution of problem (EP). Hence, since the solution of is unknown, we will use the sequence,

$$\epsilon_k = d^2(x^{k+1}, y^k) + d^2(x^k, y^k), \quad k = 0, 1, 2, \ldots$$

to study the convergence of Algorithm 1. The covergence of ϵ_k to 0 implies that the sequence $\{x^k\}$ converges to the solution of the problem. Next, figures 2, 3, 4 and 5 describe the behaviour of ϵ_k generated by Algorithm 1 for the four cases of $\{\lambda_k\}$, we have performed experiments for both number of iterations (# iteration) and elapsed execution time (Elapsed time [sec]). In these figures, the x-axes are for number of iterations or elapsed execution time while the y-axes represent for value of ϵ_k.

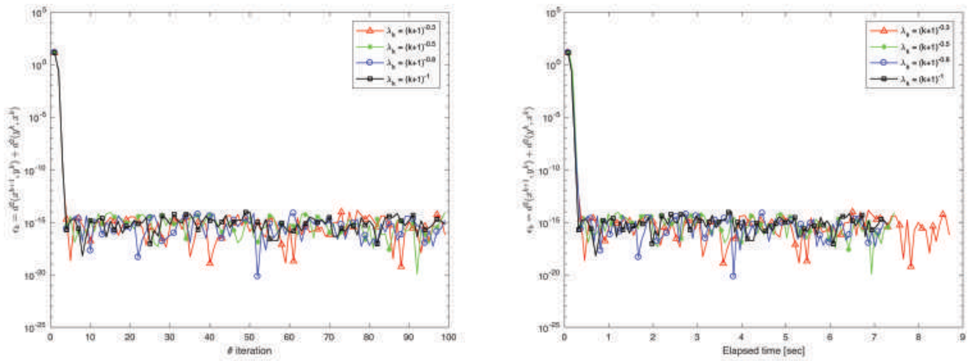

FIGURE 2. Algorithm 1 using step size class I

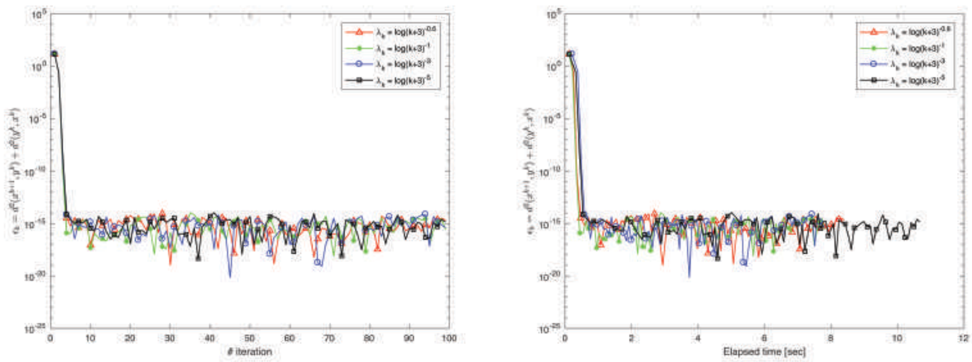

FIGURE 3. Algorithm 1 using step size class II

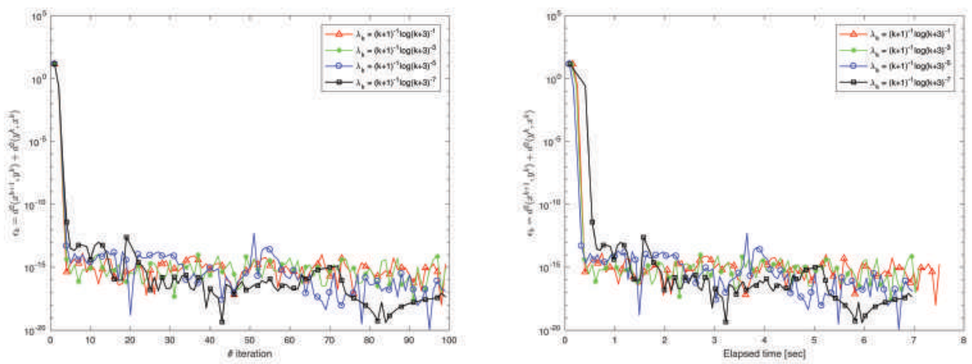

FIGURE 4. Algorithm 1 using step size class III

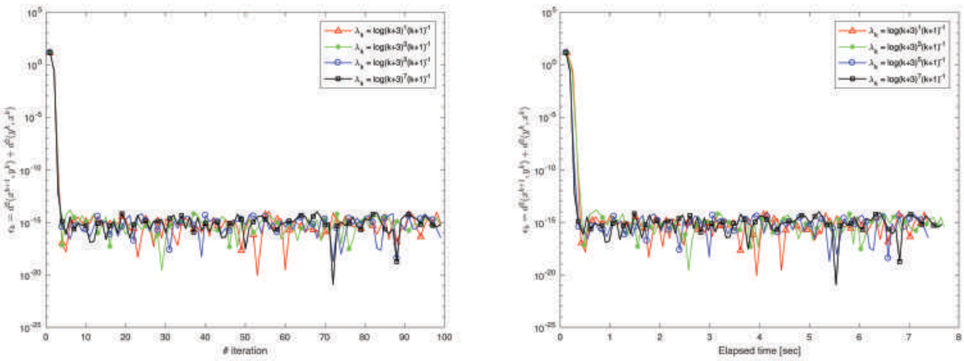

FIGURE 5. Algorithm 1 using step size class IV

From Algorithm 2, we see that if $y^k = x^k$, then x^k is the solution of problem (EP). Hence, since the solution of is unknown, we will use the sequence,

$$\epsilon_k = d(x^k, y^k), \quad k = 0, 1, 2, \ldots$$

to study the convergence of Algorithm 2. The convergence of ϵ_k to 0 implies that the sequence $\{x^k\}$ converges to the solution of the problem. Next, figures 6, 7, 8 and 9 describe the behaviour of ϵ_k generated by Algorithm 2 for the four cases of $\{\lambda_k\}$, we also performed experiments for both number of iterations (# iteration) and elapsed execution time (Elapsed time [sec]).

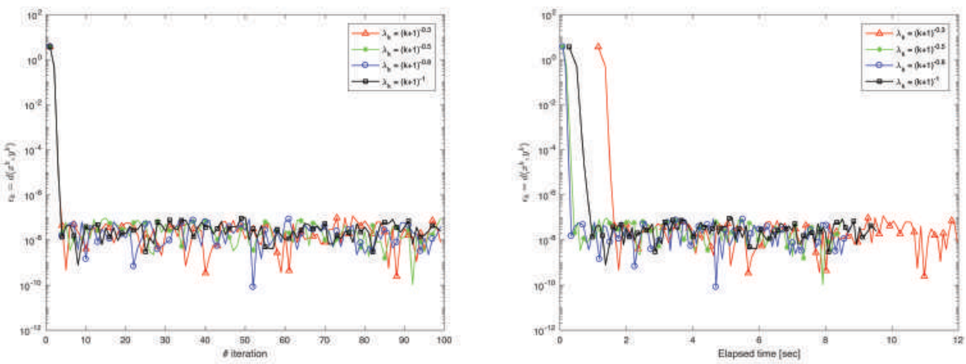

FIGURE 6. Algorithm 2 using step size class I

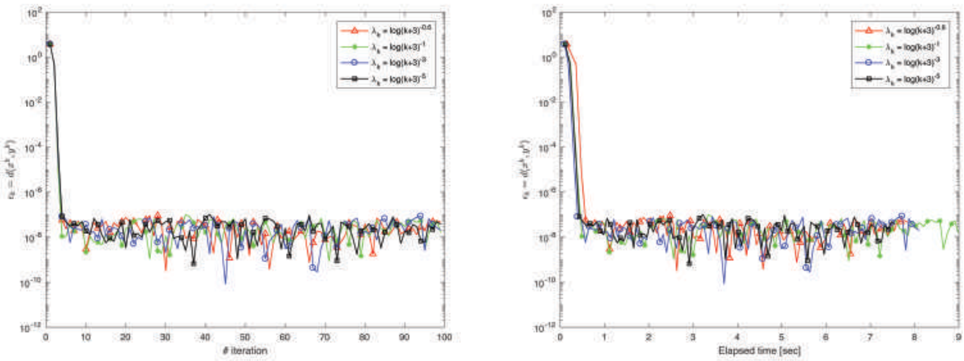

FIGURE 7. Algorithm 2 using step size class II

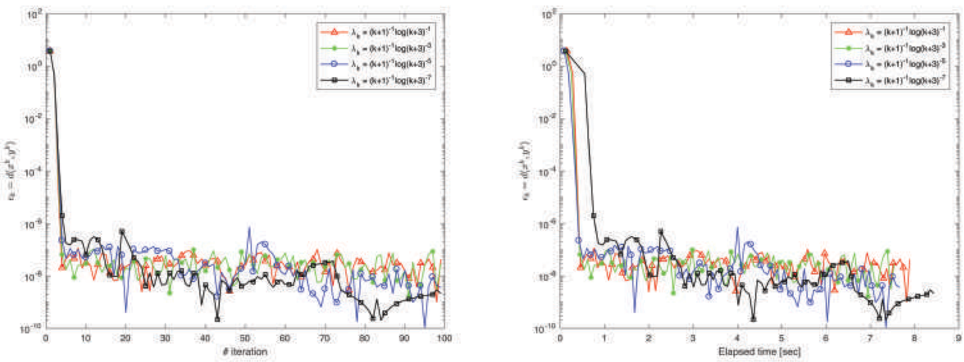

FIGURE 8. Algorithm 2 using step size class III

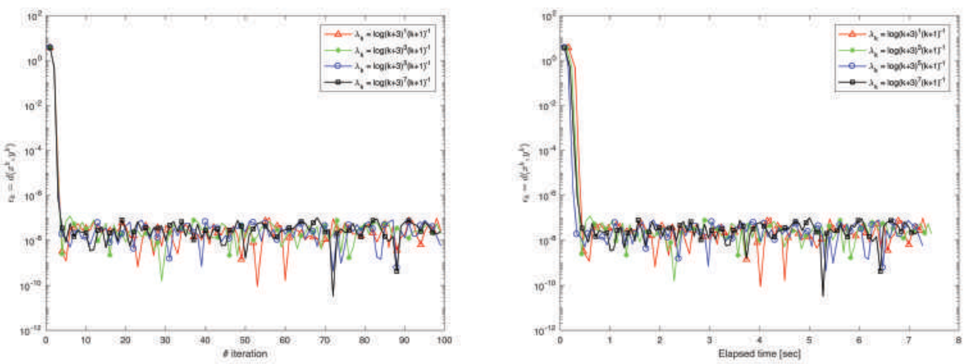

FIGURE 9. Algorithm 2 using step size class IV

To summarize from figures 2, 3, 4, 5, 6, 7, 8 and 9, we observe that the rate of convergence of $\{\epsilon_k\}$ generated by Algorithm 1 and Algorithm 2 don't depend on the convergent rate of step size sequence $\{\lambda_k\}$. For early iteration $\{\epsilon_k\}$ going to decrease quickly, but after that, it is seen to be unstable.

5. Conclusions

In this paper, we have presented two extragradient algorithms for solving a class of strongly pseudomontone and Lipschitz-type equilibrium problems in Hadamard manifolds. Under appropriate conditions, we proved the any sequences generated by the proposed algorithms converge to equilibrium points. The numerical behaviour of the extragradient algorithms on a test problem with different given stepsize sequences is also disused in this paper.

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References

[1] R. L. Adler, J.-P. Dedieu, J. Y. Margulies, M. Martens, M. Shub, Newton's method on Riemannian manifolds and a geometric model for the human spine, IMA J. Numer. Anal. 22(2002) 359–390.

[2] Q. H. Ansari, F. Babu, J.-C. Yao, Regularization of proximal point algorithms in Hadamard manifolds, J. Fixed Point Theory Appl. 21(2019) 1–23.

[3] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63(1994) 123–145.

[4] R. S. Burachik, C. Y. Kaya, M. Mammadov, An inexact modified subgradient algorithm for nonconvex optimization, Computational Optimization and Applications 45(2008) 1–24.

[5] P. Chaipunya, P. Kumam, Nonself KKM maps and corresponding theorems in hadamard manifolds, Applied General Topology 16(2015) 37.

[6] G. Cohen, Auxiliary problem principle and decomposition of optimization problems, J. Optim. Theory Appl. 32(1980) 277–305.

[7] G. Cohen, Auxiliary problem principle extended to variational inequalities, J. Optim. Theory Appl. 59(1988) 325–333.

[8] V. Colao, G. López, G. Marino, V. Martín-Márquez, Equilibrium problems in Hadamard manifolds, J. Math. Anal. Appl. 388(2012) 61–77.

[9] M. P. a. do Carmo, Riemannian geometry, Birkhäuser Boston, Inc., Boston, MA, 1992.

[10] F. Facchinei, J.-S. Pang, Finite-dimensional variational inequalities and complementarity problems, Springer-Verlag, New York, 2003.

[11] K. Fan, A minimax inequality and applications, Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin), Academic Press, New York (1972), 103–113.

[12] O. P. Ferreira, P. R. Oliveira, Proximal point algorithm on Riemannian manifolds. Optimization 51(2002) 257–270.

[13] S. D. Flå m, A. S. Antipin, Equilibrium programming using proximal-like algorithms. Math. Programming 78(1997) 29–41.

[14] D. V. Hieu, Cyclic subgradient extragradient methods for equilibrium problems, Arabian Journal of Mathematics 5(2016) 159–175.

[15] D. V. Hieu, Convergence analysis of a new algorithm for strongly pseudomontone equilibrium problems, Numer. Algorithms 77(2018) 983–1001.

[16] D. V. Hieu, New extragradient method for a class of equilibrium problems in Hilbert spaces, Appl. Anal. 97(2018) 811–824.

[17] D. V. Hieu, Y. J. Cho, Y.-b. Xiao, Modified extragradient algorithms for solving equilibrium problems, Optimization 67(2018) 2003–2029.

[18] S. Jana, C. Nahak, Mixed equilibrium problems on hadamard manifolds, Rendiconti del Circolo Matematico di Palermo 65(2015) 97–109.

[19] I. V. Konnov, Application of the proximal point method to nonmonotone equilibrium problems, J. Optim. Theory Appl. 119(2003) 317–333.

[20] I. V. Konnov, Equilibrium models and variational inequalities, Elsevier B. V., Amsterdam, 2007.

[21] G. M. Korpelevič, An extragradient method for finding saddle points and for other problems, Èkonom. i Mat. Metody 12(1976) 747–756.

[22] C. Li, G. López, V. Martín-Márquez, J.-H. Wang, Resolvents of set-valued monotone vector fields in Hadamard manifolds, Set-Valued Var. Anal. 19(2011) 361–383.

[23] W. Li, Y.-B. Xiao, N.-J. Huang, Y. J. Cho, A class of differential inverse quasi-variational inequalities in finite dimensional spaces, J. Nonlinear Sci. Appl. 10(2017) 4532–4543.

[24] B. Martinet, Régularisation d'inéquations variationnelles par approximations successives, Rev. Française Informat. Recherche Opérationnelle 4(1970) 154–158.

[25] G. Mastroeni, On auxiliary principle for equilibrium problems, Equilibrium problems and variational models (Erice, 2000), Kluwer Acad. Publ., Norwell, MA (2003) 289–298.

[26] A. Moudafi, Proximal point algorithm extended to equilibrium problems, J. Nat. Geom. 15(1999) 91–100.

[27] L. D. Muu, W. Oettli, Convergence of an adaptive penalty scheme for finding constrained equilibria, Nonlinear Anal. 18(1992) 1159–1166.

[28] J. X. C. Neto, F. M. O. Jacinto, P. A. Soares, J. C. O. Souza, On maximal monotonicity of bifunctions on hadamard manifolds, Journal of Global Optimization 72(2018) 591–601.

[29] J. X. C. Neto, P. S. M. Santos, P. A. Soares, An extragradient method for equilibrium problems on hadamard manifolds, Optimization Letters 10(2015) 1327–1336.

[30] R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optimization 14(1976) 877–898.

[31] T. Sakai, Riemannian geometry, American Mathematical Society, Providence, RI, 1996.

[32] S. T. Smith, Optimization techniques on Riemannian manifolds, Hamiltonian and gradient flows, algorithms and control, Amer. Math. Soc., Providence, RI (1994) 113–136.

[33] G.-j. Tang, N.-j. Huang, An inexact proximal point algorithm for maximal monotone vector fields on Hadamard manifolds, Oper. Res. Lett. 41(2013) 586–591.

[34] D. Q. Tran, M. L. Dung, V. H. Nguyen, Extragradient algorithms extended to equilibrium problems, Optimization 57(2008) 749–776.

[35] C. Udrişte, Convex functions and optimization methods on Riemannians manifolds, Kluwer Academic Publishers Group, Dordrecht, 1994.

[36] Y.-M. Wang, Y.-B. Xiao, X. Wang, Y. J. Cho, Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems, J. Nonlinear Sci. Appl. 9(2016) 1178–1192.