



A NEW ITERATIVE SCHEME USING INERTIAL TECHNIQUE FOR THE SPLIT FEASIBILITY PROBLEM WITH APPLICATION TO COMPRESSED SENSING

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Abstract In this work, we suggest iterative methods by using inertial term for solving split feasibility problem in real Hilbert spaces. We provide weak and strong convergence theorems under mild conditions. Finally, we apply our main result to signal recovery and give a comparison to other algorithms.

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1. INTRODUCTION

We are interested in solving the split feasibility problem (SFP) of the following form:

$$x^* \in C \text{ such that } Ax^* \in Q \quad (1.1)$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 , respectively, and $A : H_1 \rightarrow H_2$ is a bounded linear operator. This problem was proposed by Censor and Elfving [1].

In recent years, problem (1.1) draws many researchers' attention due to its wide range applications, such as matrix completion [2], image processing and compressed sensing [3–5]. The solution set of this problem will be denoted by S . Many effective methods have been proposed to solve problem (1.1). An iterative scheme for solving the split feasibility problem (SFP) is CQ algorithm which was introduced by Byrne [6, 7], whose recursive formula is

$$x_{n+1} = P_C(x_n - \tau_n A^*(I - P_Q)Ax_n) \quad (1.2)$$

where the stepsize $\tau_n \in (0, 2/\|A\|^2)$, A^* is the adjoint operator of A , P_C and P_Q are the metric projections onto C and Q , respectively.

Recently, Yang [8] introduced the following iteration:

$$x_{n+1} = P_{C_n}(x_n - \tau_n A^*(I - P_{Q_n})Ax_n) \quad (1.3)$$

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where the stepsize τ_n and A^* are similarly to that of Byrne [6, 7]. Here P_C and P_Q are replaced by P_{C_n} and P_{Q_n} , respectively, where C_n and Q_n are two half-spaces. This is noted that the closed forms of P_{C_n} and P_{Q_n} are easily calculated.

The stepsize of the iteration (1.3) is established under the assumptions that compute the operator norm of A is easily to be computed. In 2012, López et al. [9] provided the stepsize which do not depend on the operator norm:

$$\tau_n = \frac{\rho_n \|(I - P_{Q_n})Ax\|^2}{2\|A^*(I - P_{Q_n})Ax\|^2} \quad (1.4)$$

where $\{\rho_n\}$ is a sequence in $(0, 4)$ such that $\inf_{n \in \mathbb{N}} \rho_n(4 - \rho_n) > 0$. They have shown that the sequence $\{x_n\}$ generated by iteration (1.3) converges weakly to a solution of SFP. Also, they proved strongly convergence by using the Halpern-type algorithm.

In 2017, Gibali et al. [10] introduced iterative scheme with Armijo linesearches as follows:

$$\begin{aligned} y_n &= P_{C_n}(x_n - \tau_n A^*(I - P_{Q_n})Ax_n) \\ x_{n+1} &= P_{C_n}(x_n - \tau_n A^*(I - P_{Q_n})Ay_n) \end{aligned} \quad (1.5)$$

where the stepsize $\tau_n = \gamma \ell^{m_n}$ and m_n is the smallest nonnegative integer, and $\gamma > 0$, $\ell \in (0, 1)$ and $\mu \in (0, 1)$ such that

$$\tau_n \|A^*(I - P_{Q_n})x_n - A^*(I - P_{Q_n})y_n\| \leq \mu \|x_n - y_n\|. \quad (1.6)$$

They proved that $\{x_n\}$ weakly converges to the solution of SFP.

In 2019, Kesornprom et al. [11] studied the relaxed CQ algorithm with the stepsize defined by López et al [9] in Hilbert spaces as follows:

$$\begin{aligned} y_n &= x_n - \tau_n A^*(I - P_{Q_n})Ax_n \\ x_{n+1} &= P_{C_n}(y_n - \varphi_n A^*(I - P_{Q_n})y_n), \end{aligned} \quad (1.7)$$

where $\tau_n = \frac{\rho_n \|(I - P_{Q_n})Ax_n\|^2}{2\|A^*(I - P_{Q_n})Ax_n\|^2 + \beta_n}$ and $\varphi_n = \frac{\rho_n \|(I - P_{Q_n})Ay_n\|^2}{2\|A^*(I - P_{Q_n})Ay_n\|^2 + \beta_n}$. Also, to obtain the strong convergence, they introduced the following algorithm:

$$\begin{aligned} y_n &= x_n - \tau_n A^*(I - P_{Q_n})Ax_n \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)P_{C_n}(y_n - \varphi_n A^*(I - P_{Q_n})y_n) \end{aligned} \quad (1.8)$$

where $\{\alpha_n\} \subset (0, 1)$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

In this work, using idea of Nesterov [12] and Alvarez and Attouch [13], we introduce a new algorithm for solving the SFP in Hilbert spaces. We prove the convergence of the sequence generated by the proposed algorithms. Finally, we apply to compressed sensing in signal recovery and compare with algorithms of Yang [8], Gibali et al. [10] and Kesornprom et al. [11].

2. PRELIMINARIES

In this section, we give some preliminaries and lemmas which will be used in the sequel. Let H_1 and H_2 be real Hilbert spaces. Recall that a mapping $T : H_1 \rightarrow H_1$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H_1. \quad (2.1)$$

A mapping $T : H_1 \rightarrow H_1$ is said to be firmly nonexpansive if, for all $x, y \in H_1$,

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle. \quad (2.2)$$

Recall that a function $f : H_1 \rightarrow \mathcal{R}$ is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in (0, 1), \forall x, y \in H_1. \tag{2.3}$$

In a real Hilbert space H , we have the following equality:

$$\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2. \tag{2.4}$$

A differentiable function f is convex if and only if there holds the inequality:

$$f(z) \geq f(x) + \langle \nabla f(x), z - x \rangle, \quad \forall z \in H_1. \tag{2.5}$$

Recall that an element $g \in H_1$ is said to be a subgradient of $f : H_1 \rightarrow \mathcal{R}$ at x if

$$f(z) \geq f(x) + \langle g, z - x \rangle, \quad \forall z \in H_1. \tag{2.6}$$

This relation is called the subdifferentiable inequality.

A function $f : H_1 \rightarrow \mathcal{R}$ is said to be subdifferentiable at x , if it has at least one subgradient at x . The set of subgradients of f at the point x is called the subdifferentiable of f at x , and it is denoted by $\partial f(x)$. A function f is called subdifferentiable, if it is subdifferentiable at all $x \in H_1$. If a function f is differentiable and convex, then its gradient and subgradient coincide.

A function $f : H_1 \rightarrow \mathcal{R}$ is said to be weakly lower semi-continuous (w-lsc) at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n). \tag{2.7}$$

We know that the orthogonal projection of x onto C is defined as

$$P_C x := \arg \min_{y \in C} \|x - y\|^2, \quad x \in H_1. \tag{2.8}$$

By Lemma 2.2 (ii) below this is a firmly nonexpansive mapping.

Lemma 2.1. [7] *Let C and Q be closed and convex subsets of real Hilbert spaces H_1 and H_2 , respectively and $A : H_1 \rightarrow H_2$ a bounded linear operator. Let $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ then ∇f is $\|A\|^2$ -Lipschitz continuous.*

Lemma 2.2. [14] *Let C be a nonempty, closed and convex subset of a real Hilbert space H_1 . Then for any $x \in H_1$, the following assertions hold:*

- (i) $\langle x - P_C x, z - P_C x \rangle \leq 0$ for all $z \in C$;
- (ii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$ for all $x, y \in H_1$;
- (iii) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$ for all $z \in C$.

From Lemma 2.2 (ii), the operator $I - P_C$ is also firmly nonexpansive, where I denotes the identity operator, i.e., for any $x, y \in H_1$,

$$\|(I - P_C)x - (I - P_C)y\|^2 \leq \langle (I - P_C)x - (I - P_C)y, x - y \rangle. \tag{2.9}$$

Lemma 2.3. [15] *Let S be a nonempty, closed and convex subset of a real Hilbert space H_1 and $\{x_n\}$ be a sequence in H_1 that satisfies the following properties:*

- (i) $\lim_{n \rightarrow \infty} \|x_n - x\|$ exists for each $x \in S$;
- (ii) $\omega_w(x_n) \subset S$, where $\omega_w(x_n) = \{x | \exists (x_{n_k}) \subset (x_n) \text{ such that } x_{n_k} \rightharpoonup x\}$ denotes the weak ω -limit set of (x_n) .

Then $\{x_n\}$ converges weakly to a point in S .

Lemma 2.4. [16, 17] Let $\{a_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n \geq 1, \quad (2.10)$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.

(ii) If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n \rightarrow \infty} b_n / \delta_n \leq 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.5. [18] Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \delta_n, \quad n \geq 1, \quad (2.11)$$

$$s_{n+1} \leq s_n - \lambda_n + \gamma_n, \quad n \geq 1, \quad (2.12)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\gamma_n\}$ are two sequences in \mathcal{R} such that

$$(i) \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(ii) \lim_{n \rightarrow \infty} \gamma_n = 0;$$

(iii) $\lim_{k \rightarrow \infty} \lambda_{n_k} = 0$ implies $\limsup_{k \rightarrow \infty} \delta_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.6. [19] Assume $x_n \in [0, \infty)$ and $\delta_n \in [0, \infty)$ satisfy:

$$(i) \quad x_{n+1} - x_n \leq \theta_n(x_n - x_{n-1}) + \delta_n,$$

$$(ii) \quad \sum_{n=1}^{+\infty} \delta_n < \infty,$$

(iii) $\{\theta_n\} \subset [0, \theta]$, where $\theta \in [0, 1)$. Then the sequence $\{x_n\}$ is convergent with $\sum_{n=1}^{+\infty} [x_{n+1} - x_n]_+ < \infty$, where $[t]_+ := \max\{t, 0\}$ (for any $t \in \mathbb{R}$).

3. WEAK CONVERGENCE THEOREM

In this section, we introduce a new algorithm by inertial technique and prove the weak convergence theorem. In practical applications, the sets C_n and Q_n are given by

$$C_n = \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x \rangle\}, \quad (3.1)$$

where $\xi_n \in \partial c(x_n)$ and

$$Q_n = \{y \in H_2 : q(Ax_n) \leq \langle \zeta_n, Ax_n - y \rangle\}, \quad (3.2)$$

where $\zeta_n \in \partial q(Ax_n)$. In what follows, define

$$f_n(x) = \frac{1}{2} \|(I - P_{Q_n})Ax\|^2, \quad n \geq 1 \quad (3.3)$$

where Q_n is given as in (3.2). In this case, we then have

$$\nabla f_n(x) = A^*(I - P_{Q_n})Ax. \quad (3.4)$$

Algorithm 3.1. Choose an arbitrary initial guess x_1 . Given constant $\{\theta_n\} \subset [0, \theta]$ where $\theta \in [0, 1)$. Compute x_{n+1} via the formulas

$$w_n = x_n + \theta_n(x_n - x_{n-1}) \tag{3.5}$$

$$y_n = w_n - \tau_n \nabla f_n(w_n) \tag{3.6}$$

$$x_{n+1} = P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) \tag{3.7}$$

where C_n is given in (3.1), $f_n, \nabla f_n$ in (3.3) and (3.4) respectively, and

$$\tau_n = \frac{\rho_n f_n(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} \text{ and } \varphi_n = \frac{\rho_n f_n(y_n)}{\|\nabla f_n(y_n)\|^2 + \beta_n}, \quad 0 < \rho_n < 4, 0 < \beta_n < 1. \tag{3.8}$$

Theorem 3.2. Assume that $\inf_n \rho_n(4 - \rho_n) > 0, \lim_{n \rightarrow \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to a point in the solution set S .

Proof. Let $z \in S$. Since $C \subseteq C_n$ and $Q \subseteq Q_n$, we have $z = P_C(z) = P_{C_n}(z)$ and $Az = P_Q(Az) = P_{Q_n}(Az)$. It follows that $\nabla f_n(z) = 0$. Using Lemma 2.2 (iii), we see that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) - z\|^2 \\ &\leq \|y_n - \varphi_n \nabla f_n(y_n) - z\|^2 - \|x_{n+1} - y_n + \varphi_n \nabla f_n(y_n)\|^2 \\ &= \|y_n - z\|^2 + \varphi_n^2 \|\nabla f_n(y_n)\|^2 - 2\varphi_n \langle y_n - z, \nabla f_n(y_n) \rangle \\ &\quad - \|x_{n+1} - y_n + \varphi_n \nabla f_n(y_n)\|^2. \end{aligned} \tag{3.9}$$

From (2.9) and $\nabla f_n(z) = 0$, we obtain

$$\begin{aligned} \langle y_n - z, \nabla f_n(y_n) \rangle &= \langle y_n - z, \nabla f_n(y_n) - \nabla f_n(z) \rangle \\ &= \langle y_n - z, A^*(I - P_{Q_n})Ay_n - A^*(I - P_{Q_n})Az \rangle \\ &= \langle Ay_n - Az, (I - P_{Q_n})Ay_n - (I - P_{Q_n})Az \rangle \\ &\geq \|(I - P_{Q_n})Ay_n\|^2 \\ &= 2f_n(y_n). \end{aligned} \tag{3.10}$$

It also follows that

$$\langle w_n - z, \nabla f_n(w_n) \rangle \geq 2f_n(w_n). \tag{3.11}$$

Moreover, by (3.11), we see that

$$\begin{aligned} \|y_n - z\|^2 &= \|w_n - \tau_n \nabla f_n(w_n) - z\|^2 \\ &= \|w_n - z\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2 - 2\tau_n \langle w_n - z, \nabla f_n(w_n) \rangle \\ &\leq \|w_n - z\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2 - 4\tau_n f_n(w_n). \end{aligned} \tag{3.12}$$

Consider,

$$\begin{aligned} \|w_n - z\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - z\|^2 \\ &= \|x_n - z\|^2 + 2\langle x_n - z, \theta_n(x_n - x_{n-1}) \rangle + \|\theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - z\|^2 + 2\theta_n \langle x_n - z, x_n - x_{n-1} \rangle + \theta_n^2 \|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.13}$$

By (2.4), we obtain

$$\langle x_n - z, x_n - x_{n-1} \rangle = \frac{1}{2} \|x_n - z\|^2 + \frac{1}{2} \|x_n - x_{n-1}\|^2 - \frac{1}{2} \|x_{n-1} - z\|^2. \tag{3.14}$$

Combining (3.9)-(3.14), we obtain

$$\begin{aligned}
 \|x_{n+1} - z\|^2 &\leq \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|) + 2\theta_n\|x_n - x_{n-1}\|^2 \\
 &\quad + \tau_n^2\|\nabla f_n(w_n)\|^2 - 4\tau_n f_n(w_n) + \varphi_n^2\|\nabla f_n(y_n)\|^2 - 4\varphi_n f_n(y_n) \\
 &\quad - \|x_{n+1} - y_n + \varphi_n \nabla f_n(y_n)\|^2 \\
 &= \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|) + 2\theta_n\|x_n - x_{n-1}\|^2 \\
 &\quad + \frac{\rho_n^2 f_n^2(w_n)}{(\|\nabla f_n(w_n)\|^2 + \beta_n)^2} \|\nabla f_n(w_n)\|^2 - \frac{4\rho_n f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} \\
 &\quad + \frac{\rho_n^2 f_n^2(y_n)}{(\|\nabla f_n(y_n)\|^2 + \beta_n)^2} \|\nabla f_n(y_n)\|^2 - \frac{4\rho_n f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \beta_n} \\
 &\quad - \|x_{n+1} - y_n + \varphi_n \nabla f_n(y_n)\|^2 \\
 &\leq \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|) + 2\theta_n\|x_n - x_{n-1}\|^2 \\
 &\quad + \frac{\rho_n^2 f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} - \frac{4\rho_n f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} \\
 &\quad + \frac{\rho_n^2 f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \beta_n} - \frac{4\rho_n f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \beta_n} - \|x_{n+1} - y_n + \varphi_n \nabla f_n(y_n)\|^2 \\
 &= \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|) + 2\theta_n\|x_n - x_{n-1}\|^2 \\
 &\quad - \rho_n(4 - \rho_n) \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} \\
 &\quad - \rho_n(4 - \rho_n) \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \beta_n} - \|x_{n+1} - y_n + \varphi_n \nabla f_n(y_n)\|^2. \tag{3.15}
 \end{aligned}$$

Since $0 < \rho_n < 4$ and using Lemma 2.6, it implies that $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists and $\{x_n\}$ is bounded.

From (3.15), we have

$$\liminf_{n \rightarrow \infty} \rho_n(4 - \rho_n) \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} = 0, \tag{3.16}$$

which implies by our assumptions that

$$\lim_{n \rightarrow \infty} \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} = 0. \tag{3.17}$$

Since $\{\|\nabla f_n(w_n)\|\}$ is bounded by Lemma 2.1. Hence, we obtain $\lim_{n \rightarrow \infty} f_n(w_n) = \lim_{n \rightarrow \infty} \|(I - P_{Q_n})Aw_n\| = 0$. Also, we have $\lim_{n \rightarrow \infty} f_n(y_n) = \lim_{n \rightarrow \infty} \|(I - P_{Q_n})Ay_n\| = 0$

From (3.17), we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n + \varphi_n \nabla f_n(y_n)\| = 0. \tag{3.18}$$

Consider,

$$\varphi_n \|\nabla f_n(y_n)\| = \frac{\rho_n f_n(y_n)}{\|\nabla f_n(y_n)\|^2 + \theta_n} \|\nabla f_n(y_n)\| \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{3.19}$$

By (3.18) and (3.19), we get $\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0$. From (3.5) we obtain

$$\lim_{n \rightarrow \infty} \|w_n - x_n\| = 0. \tag{3.20}$$

From (3.6) and since $\tau_n \|\nabla f_n(w_n)\| \rightarrow 0$ as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \|y_n - w_n\| = 0. \tag{3.21}$$

Consider,

$$\begin{aligned} \|x_n - x_{n+1}\| &\leq \|x_n - w_n\| + \|w_n - y_n\| + \|y_n - x_{n+1}\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.22}$$

Since $\{x_n\}$ is bounded, the set $\omega_*(x_n)$ is nonempty. Let $x^* \in \omega_*(x_n)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in H_1$.

Next, we show that x^* is in S . Since $x_{n_k+1} \in C_{n_k}$, by the definition of C_{n_k} and boundedness of ∂c , it follows that

$$\begin{aligned} c(x_{n_k}) &\leq \langle \xi_{n_k}, x_{n_k} - x_{n_k+1} \rangle \\ &\leq \|\xi_{n_k}\| \|x_{n_k} - x_{n_k+1}\| \\ &\rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned} \tag{3.23}$$

where $\xi_{n_k} \in \partial c(x_{n_k})$. By the w-lsc of c , $x_{n_k} \rightharpoonup x^*$ and (3.23), we conclude that

$$c(x^*) \leq \liminf_{k \rightarrow \infty} c(x_{n_k}) \leq 0. \tag{3.24}$$

Thus $x^* \in C$.

Next, we prove that $Ax^* \in Q$. Since $P_{Q_{n_k}}(Ax_{n_k}) \in Q_{n_k}$, we have

$$\begin{aligned} q(Ax_{n_k}) &\leq \langle \eta_{n_k}, Ax_{n_k} - P_{Q_{n_k}}(Ax_{n_k}) \rangle \\ &\leq \|\eta_{n_k}\| \|Ax_{n_k} - P_{Q_{n_k}}(Ax_{n_k})\| \\ &\rightarrow 0, \text{ as } k \rightarrow \infty. \end{aligned} \tag{3.25}$$

where $\eta_{n_k} \in \partial q(Ax_{n_k})$. By the w-lsc of q and (3.19) imply that

$$q(Ax^*) \leq \liminf_{k \rightarrow \infty} q(Ax_{n_k}) \leq 0. \tag{3.26}$$

Thus, $Ax^* \in Q$. Using Lemma 2.3, we conclude that the sequence $\{x_n\}$ converges weakly to a point in S . ■

4. STRONG CONVERGENCE THEOREM

In this section, we present algorithm involving Halpern iteration for strong convergence theorem.

Algorithm 4.1. Choose an arbitrary initial guess x_1 . Given constant $\{\theta_n\} \subset [0, \theta)$ where $\theta \in [0, 1)$. Compute x_{n+1} via the formulas

$$w_n = x_n + \theta_n(x_n - x_{n-1}) \tag{4.1}$$

$$y_n = w_n - \tau_n \nabla f_n(w_n) \tag{4.2}$$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) \tag{4.3}$$

where $\{\alpha_n\} \subset (0, 1)$, C_n is given in (3.1), $f_n, \nabla f_n$ in (3.3) and (3.4) respectively, and

$$\tau_n = \frac{\rho_n f_n(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} \text{ and } \varphi_n = \frac{\rho_n f_n(y_n)}{\|\nabla f_n(y_n)\|^2 + \beta_n}, \quad 0 < \rho_n < 4, 0 < \beta_n < 1. \tag{4.4}$$

Theorem 4.2. Assume that $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\theta_n\}$ satisfy the assumptions:

- (a1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (a2) $\inf_n \rho_n(4 - \rho_n) > 0$;
- (a3) $\lim_{n \rightarrow \infty} \beta_n = 0$.
- (a4) $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ generated by Algorithm 4.1 converges strongly to a point P_{S_u} in the solution set S .

Proof. We set $z = P_{S_u}$. Using the proof line as in Theorem 1, we obtain

$$\begin{aligned} & \|P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) - z\|^2 \\ & \leq \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n \|x_n - x_{n-1}\|^2 \\ & \quad - \rho_n(4 - \rho_n) \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} - \rho_n(4 - \rho_n) \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \beta_n} \\ & \quad - \|P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) - y_n + \varphi_n \nabla f_n(y_n)\|^2 \end{aligned} \tag{4.5}$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 & = \|\alpha_n(u - z) + (1 - \alpha_n)(P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) - z)\|^2 \\ & \leq (1 - \alpha_n) \|P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z \rangle. \end{aligned} \tag{4.6}$$

Combining (4.5)-(4.6), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 & \leq (1 - \alpha_n) \|x_n - z\|^2 + (1 - \alpha_n) \theta_n (\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\ & \quad + 2(1 - \alpha_n) \theta_n \|x_n - x_{n-1}\|^2 - (1 - \alpha_n) \rho_n(4 - \rho_n) \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} \\ & \quad - (1 - \alpha_n) \rho_n(4 - \rho_n) \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \beta_n} \\ & \quad - (1 - \alpha_n) \|P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) - y_n + \varphi_n \nabla f_n(y_n)\|^2 \\ & \quad + 2\alpha_n \langle u - z, x_{n+1} - z \rangle. \end{aligned} \tag{4.7}$$

From (4.5) and $\rho_n \in (0, 4)$, we have

$$\|P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) - z\| \leq \|w_n - z\|. \tag{4.8}$$

On the other hand, we have

$$\begin{aligned} \|w_n - z\| & = \|x_n + \theta_n(x_n - x_{n-1}) - z\| \\ & \leq \|x_n - z\| + \theta_n \|x_n - x_{n-1}\| \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9), we get

$$\begin{aligned} \|x_{n+1} - z\| & = \|\alpha_n u + (1 - \alpha_n) P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) - z\| \\ & \leq \alpha_n \|u - z\| + (1 - \alpha_n) \|w_n - z\| \\ & \leq \alpha_n \|u - z\| + (1 - \alpha_n) \|x_n - z\| + (1 - \alpha_n) \theta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{4.10}$$

By (a4), we see that $\phi = \frac{(1 - \alpha_n) \theta_n \|x_n - x_{n-1}\|}{\alpha_n} \rightarrow 0$. Hence it is bounded. Set

$$M = \max\{\|u - z\|, \sup_{n \geq 1} \phi\}. \tag{4.11}$$

So, (4.10) be comes

$$\|x_{n+1} - z\| \leq (1 - \alpha_n) \|x_n - z\| + \alpha_n M \tag{4.12}$$

By Lemma 2.4(i), we can conclude that $\{x_n\}$ is bounded. Using Lemma 2.5 and 2.6, from (4.7), we have

$$\begin{aligned}
 s_n &= \|x_n - z\|^2; \\
 \gamma_n &= (1 - \alpha_n)\theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2(1 - \alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\
 &\quad + 2\alpha_n\langle u - z, x_{n+1} - z \rangle; \\
 \delta_n &= 2\langle u - z, x_{n+1} - z \rangle; \\
 \lambda_n &= (1 - \alpha_n)\rho_n(4 - \rho_n)\frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2 + \beta_n} \\
 &\quad + (1 - \alpha_n)\rho_n(4 - \rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \beta_n} \\
 &\quad + (1 - \alpha_n)\|P_{C_n}(y_n - \varphi_n \nabla f_n(y_n)) - y_n + \varphi_n \nabla f_n(y_n)\|^2.
 \end{aligned} \tag{4.13}$$

So (4.7) reduces to the inequalities

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\delta_n, n \geq 1 \tag{4.14}$$

$$s_{n+1} \leq s_n - \lambda_n + \gamma_n. \tag{4.15}$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose that

$$\lim_{k \rightarrow \infty} \lambda_{n_k} = 0. \tag{4.16}$$

Then we have

$$\begin{aligned}
 &\lim_{k \rightarrow \infty} (1 - \alpha_{n_k})\rho_{n_k}(4 - \rho_{n_k})\frac{f_{n_k}^2(w_{n_k})}{\|\nabla f_{n_k}(w_{n_k})\|^2 + \beta_{n_k}} \\
 &+ (1 - \alpha_{n_k})\rho_{n_k}(4 - \rho_{n_k})\frac{f_{n_k}^2(y_{n_k})}{\|\nabla f_{n_k}(y_{n_k})\|^2 + \beta_{n_k}} \\
 &+ (1 - \alpha_{n_k})\|P_{C_{n_k}}(y_{n_k} - \varphi_{n_k} \nabla f_{n_k}(y_{n_k})) - y_{n_k} + \varphi_{n_k} \nabla f_{n_k}(y_{n_k})\|^2 = 0
 \end{aligned} \tag{4.17}$$

which implies, by our assumptions

$$\begin{aligned}
 &\frac{f_{n_k}^2(w_{n_k})}{\|\nabla f_{n_k}(w_{n_k})\|^2} \rightarrow 0, \frac{f_{n_k}^2(y_{n_k})}{\|\nabla f_{n_k}(y_{n_k})\|^2} \rightarrow 0 \text{ and} \\
 &\|P_{C_{n_k}}(y_{n_k} - \varphi_{n_k} \nabla f_{n_k}(y_{n_k})) - y_{n_k} + \varphi_{n_k} \nabla f_{n_k}(y_{n_k})\| \rightarrow 0
 \end{aligned}$$

as $k \rightarrow \infty$. Since $\{\|\nabla f_{n_k}(w_{n_k})\|\}$ and $\{\|\nabla f_{n_k}(y_{n_k})\|\}$ are bounded, it follows that $f_{n_k}(w_{n_k}) \rightarrow 0$ and $f_{n_k}(y_{n_k}) \rightarrow 0$ as $k \rightarrow \infty$. So we get $\lim_{k \rightarrow \infty} \|(I - P_{Q_{n_k}})Aw_{n_k}\| = 0$ and $\lim_{k \rightarrow \infty} \|(I - P_{Q_{n_k}})Ay_{n_k}\| = 0$.

As the same proof in Theorem 1, we can show that $\omega_*(x_{n_k}) \subset S$. Hence there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \rightarrow x^* \in S$.

From Lemma 2.2 (i), we obtain

$$\begin{aligned}
 \limsup_{k \rightarrow \infty} \langle u - z, x_{n_k} - z \rangle &= \lim_{i \rightarrow \infty} \langle u - z, x_{n_{k_i}} - z \rangle \\
 &= \langle u - z, x^* - z \rangle \\
 &\leq 0.
 \end{aligned} \tag{4.18}$$

On the other hand, we see that

$$\begin{aligned}
 & \|x_{n_k+1} - y_{n_k}\| \\
 = & \|\alpha_{n_k}u + (1 - \alpha_{n_k})P_{C_{n_k}}(y_{n_k} - \varphi_{n_k}\nabla f_{n_k}(y_{n_k})) - y_{n_k}\| \\
 \leq & \alpha_{n_k}\|u - y_{n_k}\| + (1 - \alpha_{n_k})\|P_{C_{n_k}}(y_{n_k} - \varphi_{n_k}\nabla f_{n_k}(y_{n_k})) - y_{n_k}\| \\
 \leq & \alpha_{n_k}\|u - y_{n_k}\| + (1 - \alpha_{n_k})\|P_{C_{n_k}}(y_{n_k} - \varphi_{n_k}\nabla f_{n_k}(y_{n_k})) - y_{n_k} + \varphi_{n_k}\nabla f_{n_k}(y_{n_k})\| \\
 & + (1 - \alpha_{n_k})\varphi_{n_k}\|\nabla f_{n_k}(y_{n_k})\| \\
 \rightarrow & 0 \text{ as } k \rightarrow \infty.
 \end{aligned}
 \tag{4.19}$$

So, we have

$$\begin{aligned}
 \|x_{n_k+1} - x_{n_k}\| & \leq \|x_{n_k+1} - y_{n_k}\| + \|y_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \\
 & \rightarrow 0 \text{ as } k \rightarrow \infty.
 \end{aligned}
 \tag{4.20}$$

From (4.18) and (4.20) we obtain

$$\limsup_{k \rightarrow \infty} \langle u - z, x_{n_k+1} - z \rangle \leq 0.
 \tag{4.21}$$

Hence, we get

$$\limsup_{k \rightarrow \infty} \delta_{n_k} \leq 0.
 \tag{4.22}$$

Using Lemma 2.5, we conclude that the sequence $\{x_n\}$ converges strongly to $z = P_S u$. ■

5. NUMERICAL EXPERIMENTS

In this section, we provide some numerical experiments in compressed sensing. In signal recovery, compressed sensing can be modeled as the following under determinated linear equation system:

$$y = Ax + \varepsilon,
 \tag{5.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε , and $A : \mathbb{R}^N \rightarrow \mathbb{R}^M$ ($M < N$) is a bounded linear observation operator. It is known that problem (5.1) can be seen as solving the following LASSO problem [20]

$$\min_{x \in \mathbb{R}^N} \frac{1}{2} \|y - Ax\|_2^2 \text{ subject to } \|x\|_1 \leq t,
 \tag{5.2}$$

where $t > 0$ is a given constant. In particular, if $C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\}$ and $Q = \{y\}$, then the LASSO problem (5.2) can be considered as the SFP (1.1).

The sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-2, 2]$ with m nonzero elements. The matrix $A \in \mathbb{R}^{M \times N}$ is generated from a normal distribution with mean zero and variance one. The observation y is generated by white Gaussian noise with signal-to-noise ratio SNR=40. The process is started with $t = m$ and let x_1 and x_0 be randomly.

The restoration accuracy is measured by the mean squared error as follows:

$$\text{MSE} = \frac{1}{N} \|x_n - x\|_2^2 < 10^{-4},$$

where x_n is an estimated signal of x .

Let $\tau_n = \frac{1}{\|A\|^2}$ in an iteration of Yang [8] and τ_n as (1.6) in an iteration of Gibali et al. [10]. We choose the parameters $\rho_n = 3.5$, $l = \mu = 0.5$. and

$$\theta_n = \begin{cases} \min \left\{ \frac{1}{n^{1.1} \|x_n - x_{n-1}\|^2}, \theta \right\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise.} \end{cases}$$

The programme is implemented in MATLAB 2018b. We consider two cases as follows:

Case 1: $M = 256, N = 512, m = 10$

Case 2: $M = 2048, N = 4096, m = 100$.

Then the numerical results are reported as follows:

TABLE 1. Computational results for weak convergence theorem

Method	Case1		Case2	
	CPU	Iter	CPU	Iter
Algorithm of Yang [8]	1.0991	69	535.6640	76
Algorithm of Gibali et al. [10]	1.7102	84	953.0753	94
Algorithm of Kesornprom et al. [11]	0.0314	43	1.8135	42
Algorithm 3.1	0.0302	32	1.6537	37

Next, we present the comparison of Algorithm 4.1 and Algorithm of Kesornprom et al. [11]. We set the parameter $\alpha_n = \frac{1}{n+1}$, x_1, x_0 and u are generated randomly. Then we have the results as follow:

TABLE 2. Computational results for strong convergence theorem

Method	Case1		Case2	
	CPU	Iter	CPU	Iter
Algorithm of Kesornprom et al. [11]	0.0088	39	1.7686	48
Algorithm 4.1	0.0086	30	1.3352	36

We plot the graphs of original signal and recovered signal.

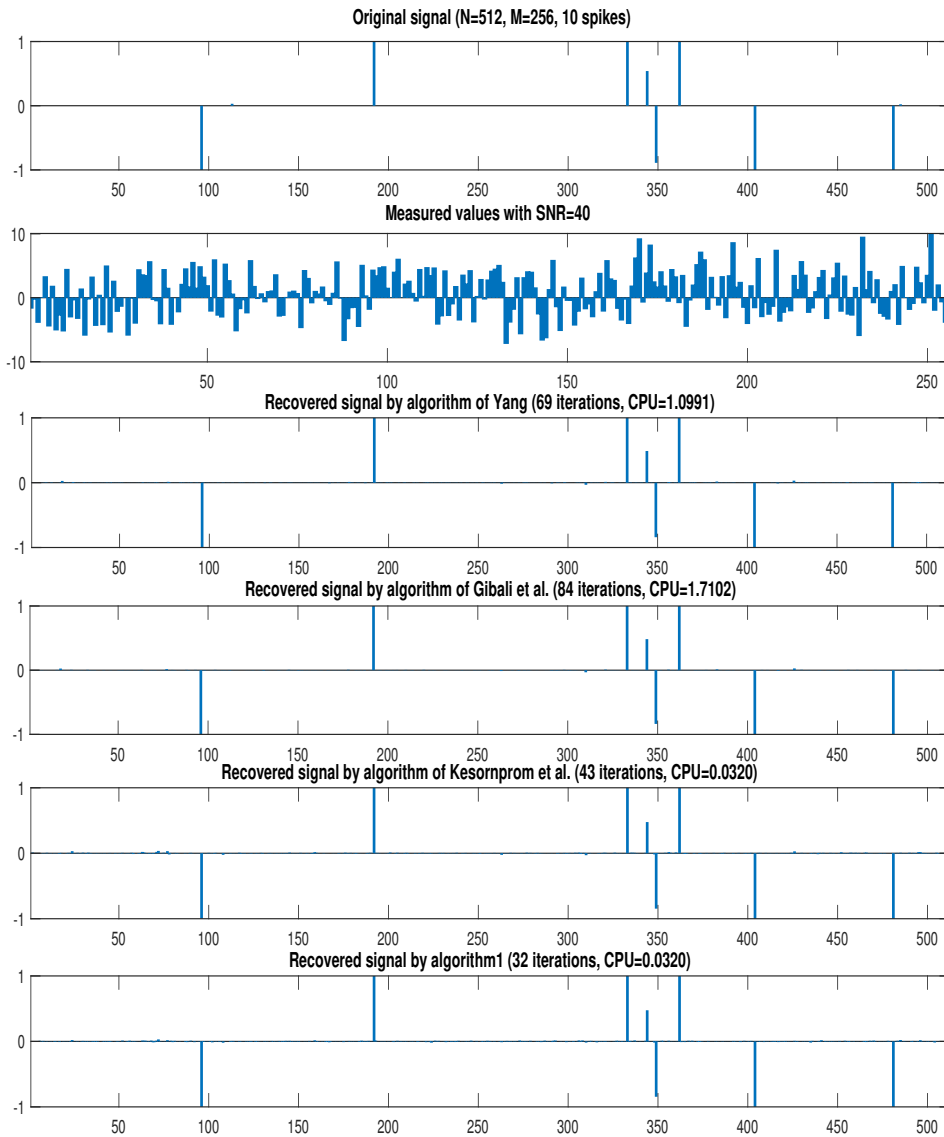


FIGURE 1. From top to bottom: original signal, observation data, recovered signal by Algorithm of Yang [8], Algorithm of Gibali et al. [10], Algorithm of Kesornprom et al. [11] and Algorithm 3.1 in Case 1

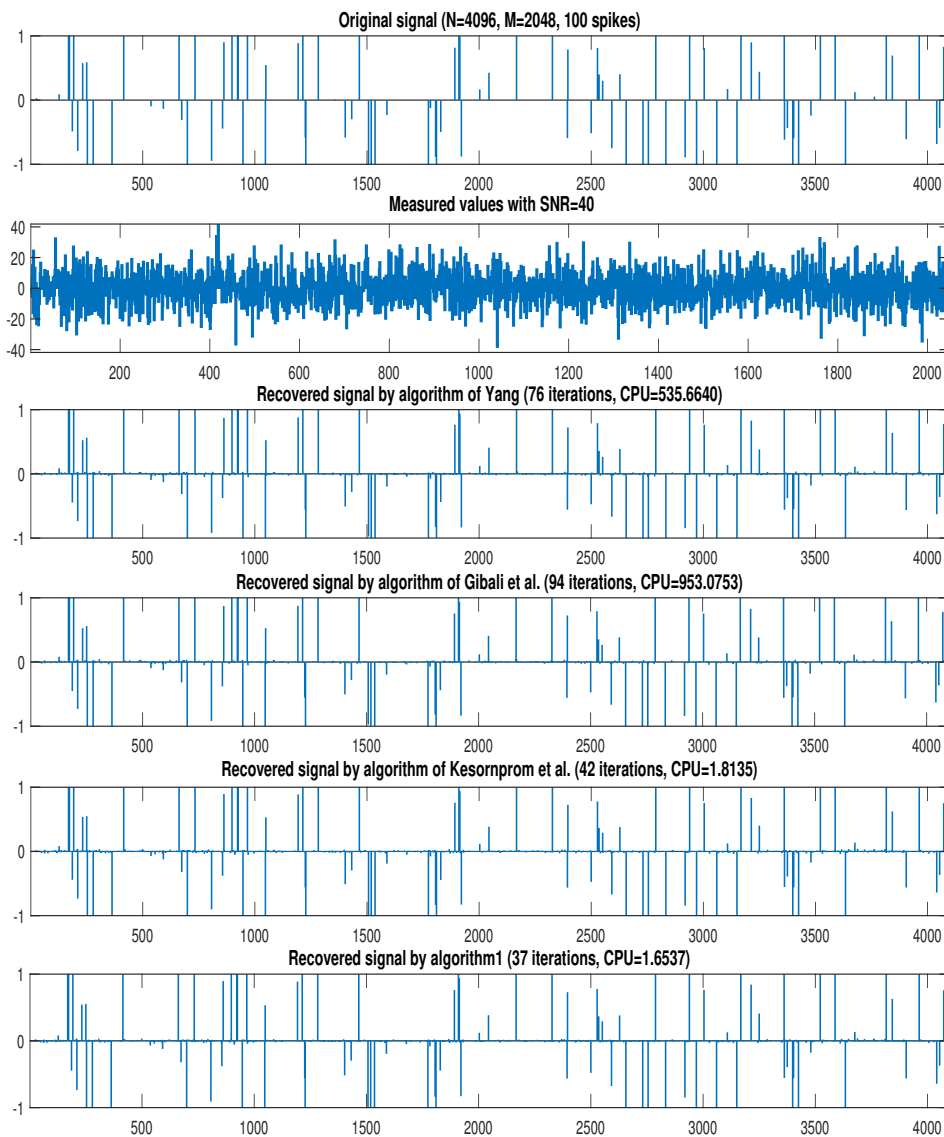


FIGURE 2. From top to bottom: original signal, observation data, recovered signal by Algorithm of Yang [8], Algorithm of Gibali et al. [10], Algorithm of Kesornprom et al. [11] and Algorithm 3.1 in Case 2

The error plotting of Algorithm of Yang [8], Algorithm of Gibali et al. [10], Algorithm of Kesornprom et al. [11] and Algorithm 3.1 is shown as follows:

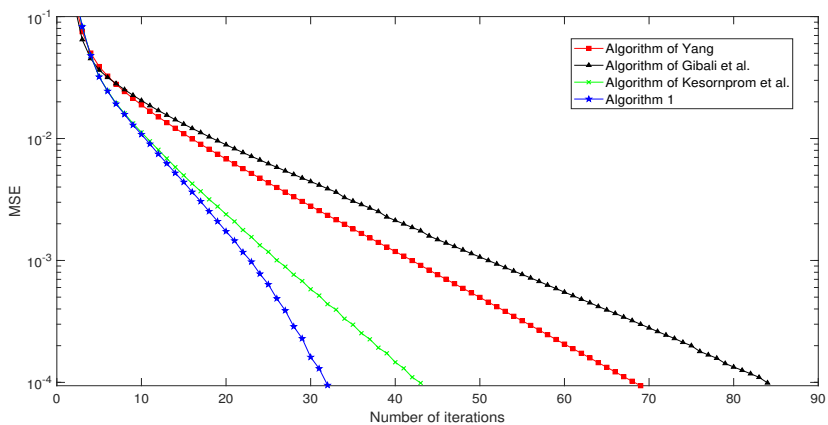


FIGURE 3. MSE versus number of iterations in case 1

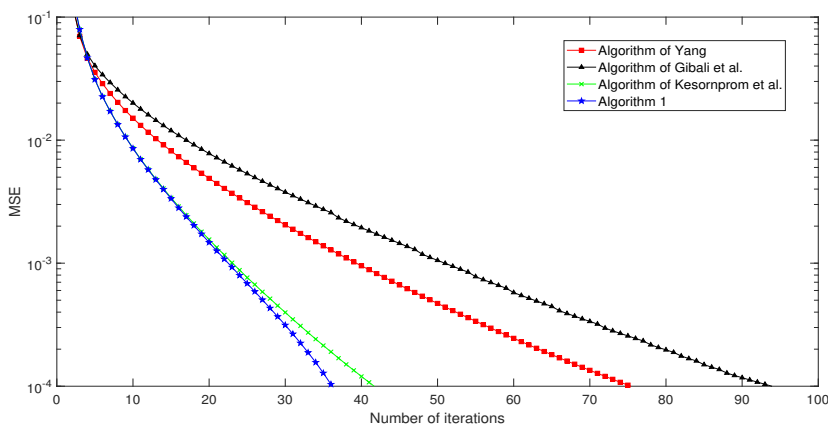


FIGURE 4. MSE versus number of iterations in case 2

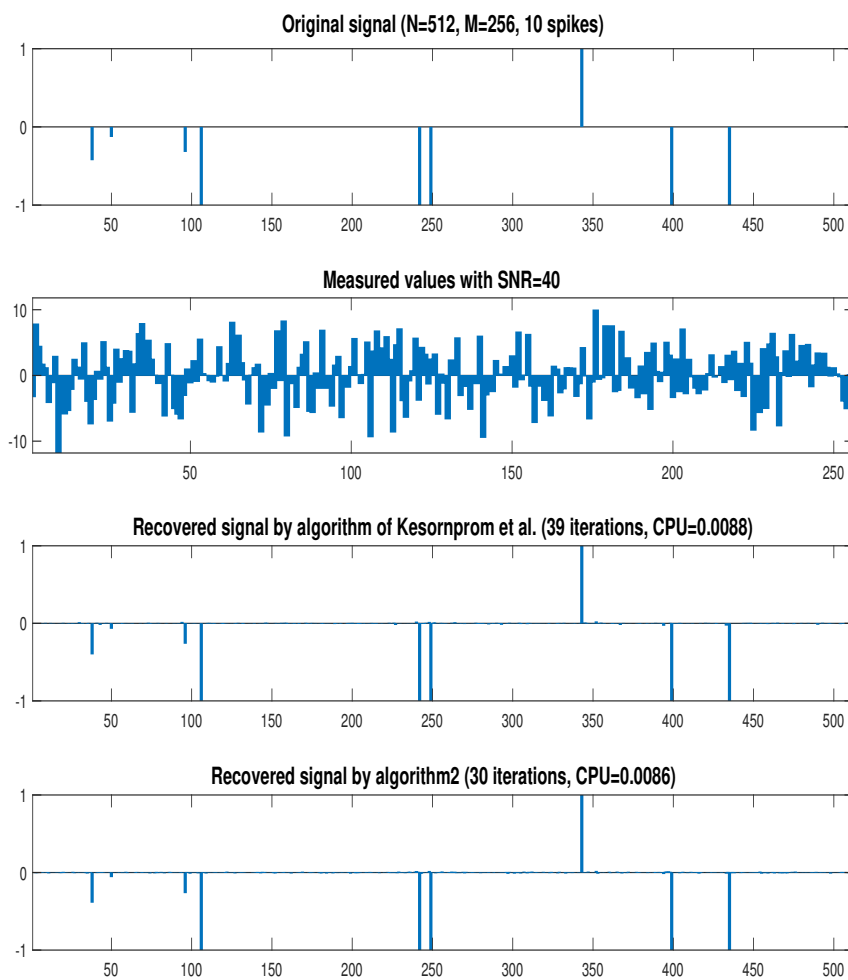


FIGURE 5. From top to bottom: original signal, observation data, recovered signal by Algorithm of Kesornprom et al. [11] and Algorithm 4.1 in Case 1

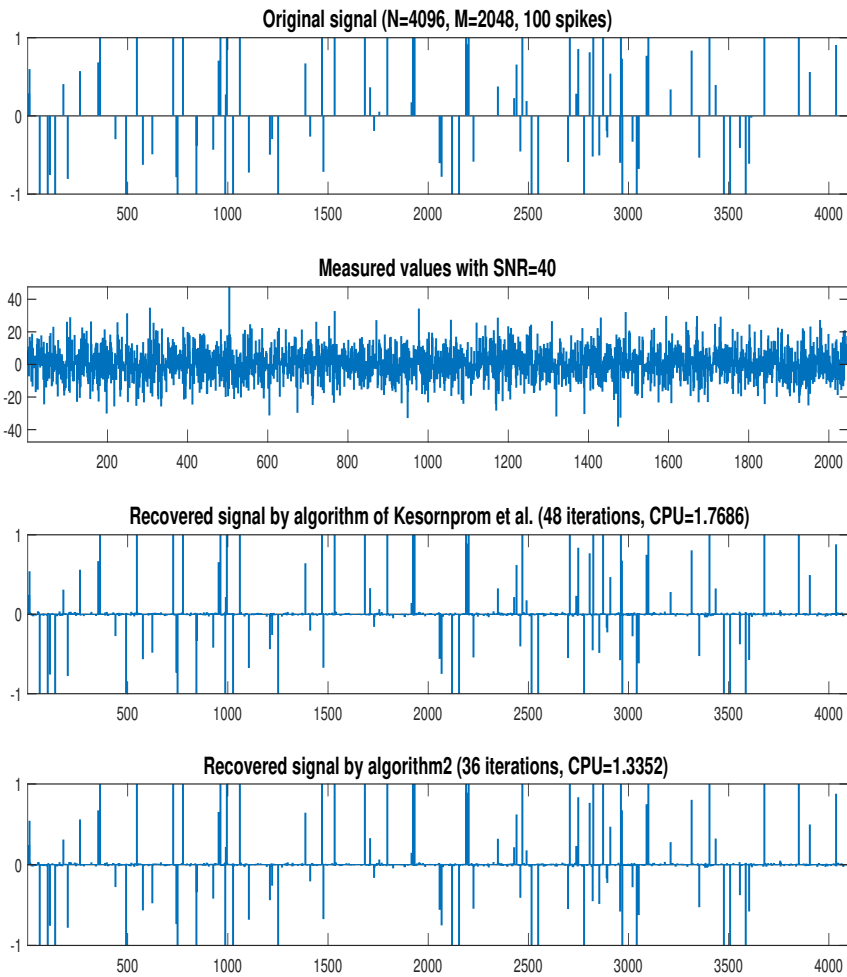


FIGURE 6. From top to bottom: original signal, observation data, recovered signal by [11] and Algorithm 4.1 in Case 2

The error plotting of Algorithm of Kesornprom et al. [11] and Algorithm 4.1 is shown as follows:

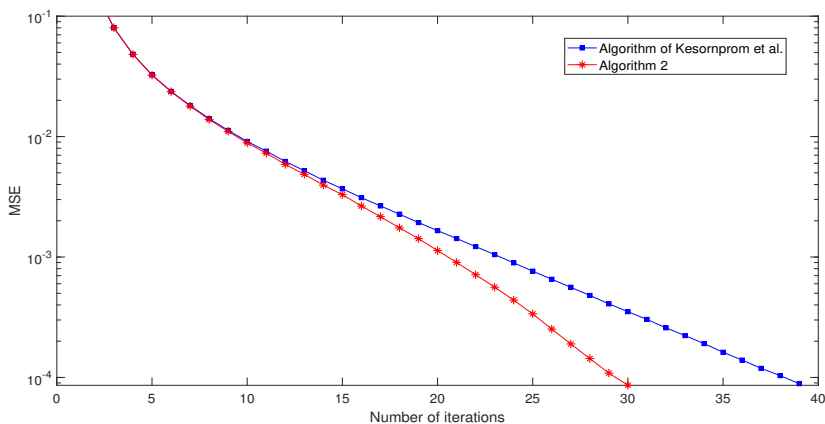


FIGURE 7. MSE versus number of iterations in case 1

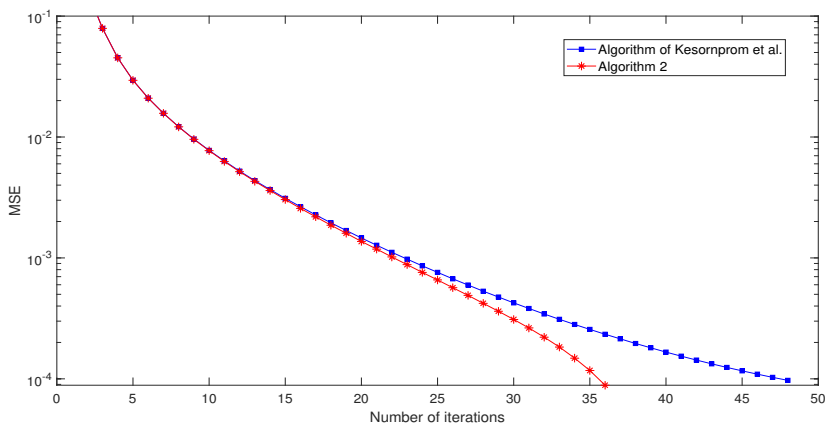


FIGURE 8. MSE versus number of iterations in case 2

6. CONCLUSIONS

In Table 1 and 2, we see that the convergence of the sequence generated by Algorithm 3.1 and Algorithm 4.1 have number of iterations and CPU time less than Algorithm of Yang [8], Algorithm of Gibali et al. [10], Algorithm of Kesornprom et al. [11], it shows that our algorithms have a better convergence than other algorithms. From Figure 1-8, we can apply our methods to recover the original signal.

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