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On Exact Solutions of Wind-driven Flow in Shallow Off-shore Waters with Nonlinear Bottom Stress

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Abstract : Complicated numerical models are sometimes used to analyze one-dimensional wind drift problems that incorporate depth-dependent viscosity, Coriolis effects, and quadratic bottom stress. This work describes a method for finding exact solutions of this type of problem. An exact depth-dependent steady-state solution should serve as a useful test of the accuracy and convergence of the solutions obtained by numerical methods. The method for obtaining exact solutions is illustrated with examples.

1 Introduction

In 2000, Bowers, Winter and Lund [1] presented the Sinc-Galerkin method as a new and potentially useful extension of the spectral method in numerical oceanography. They illustrated the technique by using a Sinc-Galerkin procedure to infer the sensitivity of wind-driven subsurface currents in coastal regions and semi-enclosed seas when the vertical eddy viscosity coefficient is a continuously differentiable function of depth [1]. In 2004, Koonprasert and Bowers [2] developed a block matrix formulation for the Sinc-Galerkin technique and applied their formulation to the wind-driven ocean current problem. Koonprasert and Bowers [3] also developed a fully Sinc-Galerkin method for solving a family of complex-valued partial differential equations with time-dependent boundary conditions. The approach to constructing an exact solution of a model of the wind-driven ocean-current problem is based on a complex velocity representation. When general solutions of the classical differential equations governing the horizontal wind drift current components are known, the boundary conditions constitute a nonlinear algebraic system whose solution provides values of four constants of integration. In the boundary condition system, one of the two complex integration constants F, is expressed in polar form. The other complex constant is eliminated in such a way as to preserve the amplitude and phase of F. The amplitude of F is found as the smallest positive root r of a quartic polynomial whose coefficients depend on the derivatives of the fundamental solutions at the boundaries. Once r is determined, all required quantities become known and the solution of the original problem is assembled by substitutions.

2 Problem formulation and nondimensionalization

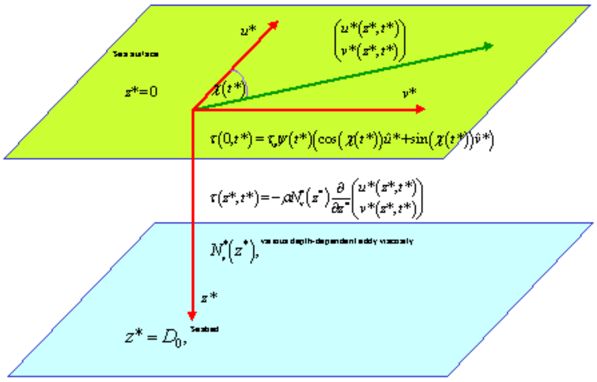

Figure 1: A picture showing the model and coordinate system

A picture of the model and coordinates used is given in Figure 1. A vertical coordinate is directed positive downward from the sea surface to the seabed. The velocities $u^*(z^*, t^*)$ and $v^*(z^*, t^*)$ are directed northward and eastward respectively. We assume that the ocean depth D_0 and mass density ρ are constant and that the effect of tides can be neglected. The function $\psi(t^*)$ represents the sea surface wind stress.

Ocean currents are driven by a time-dependent tangential surface wind stress of magnitude $\tau_w \psi(t^*)$ at surface $(z^* = 0)$ represented by

$$\tau(0, t^*) = \tau_w \psi(t^*) \begin{bmatrix} \cos(\chi(t^*)) \\ \sin(\chi(t^*)) \end{bmatrix}$$

$\chi(t^*)$ is the angle between the positive u^*-axis and the wind direction. The horizontal stress at various depths for a depth-dependent eddy viscosity $N_v^*(z^*)$, $t^* \in [0, \infty)$, $N_0 \equiv N_v^*(0)$ is:

$$\tau(z^*, t^*) = -\rho N_v^*(z^*) \frac{\partial}{\partial z^*} \begin{pmatrix} u^*(z^*, t^*) \\ v^*(z^*, t^*) \end{pmatrix}$$

By the conservation of linear momentum, there is a balance between the Coriolis force and the internal friction associated with turbulence. Thus the wind-drift current velocity $\begin{pmatrix} u^*(z^*, t^*) \\ v^*(z^*, t^*) \end{pmatrix}$ can be determined by solving the initial-boundary-value problem

$$\frac{\partial}{\partial t^*}\left(\begin{array}{c} u^*(z^*,t^*) \\ v^*(z^*,t^*) \end{array}\right) - \frac{\partial}{\partial z^*}\left[N_v^*(z^*)\frac{\partial}{\partial z^*}\left(\begin{array}{c} u^*(z^*,t^*) \\ v^*(z^*,t^*) \end{array}\right)\right] = f\left(\begin{array}{c} u^*(z^*,t^*) \\ v^*(z^*,t^*) \end{array}\right) \tag{2.1}$$

where $0 < z^* < D_0$, $0 < t^*$. The stress condition at the sea surface is equal to the tangential surface time-dependent wind stress given by:

$$-\rho N_v^*(0)\frac{\partial}{\partial z^*}\left(\begin{array}{c} u^*(0,t^*) \\ v^*(0,t^*) \end{array}\right) = \tau_w \psi(t)\left(\begin{array}{c} \cos(\chi(t)) \\ \sin(\chi(t)) \end{array}\right), 0 < t^* \tag{2.2}$$

At the seabed the frictional stress is assumed linearly proportional to the current

$$-\rho N_v^*(D_0)\frac{\partial}{\partial z^*}\left(\begin{array}{c} u^*(D_0,t^*) \\ v^*(D_0,t^*) \end{array}\right) = k_f\rho\left(\begin{array}{c} u^*(D_0,t^*) \\ v^*(D_0,t^*) \end{array}\right), 0 < t^* \tag{2.3}$$

Initially the sea is assumed to be at rest, so that

$$\left(\begin{array}{c} u^*(z^*,0) \\ v^*(z^*,0) \end{array}\right) = 0 \quad , \quad 0 < z^* < D_0$$

Nondimensionalization begins by assigning a reference value to the kinematic eddy viscosity $N_0^*\left(m^2 s^{-1}\right)$ and defining $N_v^* = N_0^* N_v$. The constant N_0^* may be regarded as the viscosity of the laminar sublayer at the seabed. Dimensionless depth and current are defined by

$$\left(\begin{array}{c} U \\ V \end{array}\right) = \frac{1}{U_0}\left(\begin{array}{c} U^* \\ V^* \end{array}\right), z = \frac{z^*}{D_0}, U_0 = \frac{\tau_w D_E}{\rho N_0^*}\left(m \cdot s^{-1}\right), D_E = \sqrt{\frac{2N_0^*}{f}}\,(m) \tag{2.4}$$

The following dimensionless parameters are also useful.

$$\kappa = \frac{D_0}{D_E} \quad , \quad k_f \approx 0.005 \quad , \quad \sigma_f = \frac{N_0^*}{k_f U_0 D_0} \tag{2.5}$$

For a steady-state solution we have $\frac{\partial}{\partial t^*}\left(\begin{array}{c} u^*(z^*,t^*) \\ v^*(z^*,t^*) \end{array}\right) = 0$, and then the equation (2.1) leads to

$$-\frac{\partial}{\partial z^*}\left[N_v^*(z^*)\frac{\partial}{\partial z^*}\left(\begin{array}{c} u^*(z) \\ v^*(z) \end{array}\right)\right] = f\left(\begin{array}{c} -v^*(z) \\ u^*(z) \end{array}\right).$$

The nondimensional problem takes the form

$$-\frac{1}{D_0}\frac{d}{dz}\left[N_0^* N_v(z)\frac{d}{D_0 dz}U_0\left(\begin{array}{c} U(z) \\ V(z) \end{array}\right)\right] = fU_0\left(\begin{array}{c} -V(z) \\ U(z) \end{array}\right)$$

$$-\frac{N_0^* U_0}{D_0^2}\frac{d}{dz}\left[N_v(z)\frac{d}{dz}\left(\begin{array}{c} U(z) \\ V(z) \end{array}\right)\right] = fU_0\left(\begin{array}{c} -V(z) \\ U(z) \end{array}\right)$$

We can also transform the velocity $\begin{pmatrix} U(z) \\ V(z) \end{pmatrix}$ into the complex velocity surrogate

$$W(z) = iU(z) + V(z) \text{ and } -iW(z) = -iV(z) + U(z)$$

to obtain the complex equation:

$$-\frac{N_0^*}{D_0^2}\frac{d}{dz}\left[N_v(z)\frac{dW(z)}{dz}\right] = -ifW(z).$$

From equations (2.4) and (2.5), by substitution, we then obtain

$$-\frac{N_0^*}{D_0^2}\frac{d}{dz}\left[N_v(z)\frac{dW(z)}{dz}\right] = -\frac{2iN_0^*\kappa^2}{D_0^2}W(z)$$
$$-\frac{d}{dz}\left[N_v(z)\frac{dW(z)}{dz}\right] = -2i\kappa^2 W(z)$$

Since $i = e^{i\frac{\pi}{2}}$, we can rewrite this last equation as

$$\frac{d}{dz}\left[N_v(z)\frac{dW(z)}{dz}\right] - 2e^{i\frac{\pi}{2}}\kappa^2 W(z) = 0 \quad , \quad 0 < z < 1. \tag{2.6}$$

Then equation (2.2) can be written as

$$N_s\frac{d}{dz}W(0) = -\kappa\begin{pmatrix} \cos(\chi) \\ \sin(\chi) \end{pmatrix}$$

where $N_v(0) = N_s$, and $\kappa = \frac{\tau_w D_0}{\rho N_0^* U_0}$. The boundary condition at the sea surface is then

$$N_s\frac{d}{dz}W(0) = -\kappa e^{i\left(\frac{\pi}{2}-\chi\right)} \tag{2.7}$$

Similarly, the boundary condition at the seabed becomes

$$\sigma_f N_b\frac{d}{dz}W(1) = -r_0 W(1)\sqrt{W(1)\,\overline{W}(1)} \tag{2.8}$$

Equations (2.6), (2.7) and (2.8) then give the boundary value problem to be solved for wind-driven ocean currents.

3 The boundary condition system

When a fundamental solution set of equation (2.6) is found for a given $N_v(z)$, it will generally be the result of a transformation $W(z) = Y(x), x = x(z)$, that carries equation (2.6) into integrable form on $x \in (x_0, x_1)$, where $x_0 = x(0)$ and $x_1 = x(1)$. The transformed conditions constitute a nonlinear algebraic system for complex constants C and D in the solution

$$W(z) = CY_1(x) + DY_2(x) \tag{3.1}$$

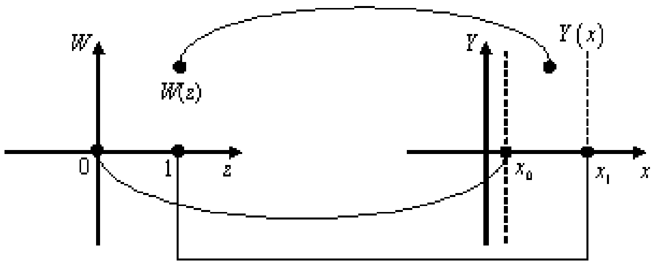

Figure 2: The transformation

It is shown by the following Figure 2
At $z = 0$ (sea surface)→ $x_0 = x(0)$, we get

$$N_s \left(\frac{dx}{dz}\right)_{x=x_0} \left[CY_1'(x) + DY_2'(x)\right] = -\kappa e^{i\left(\frac{\pi}{2}-\chi\right)} \tag{3.2}$$

At $z = 1$ (seabed) → $x_1 = x(1)$

$$\sigma_f N_b \left(\frac{dx}{dz}\right)_{x=x_1} \left[CY_1'(x) + DY_2'(x)\right] = -r_0 \left[CY_1(x) + DY_2(x)\right] \left|CY_1(x) + DY_2(x)\right| \tag{3.3}$$

Since our goal is an expression for the amplitude of one of the constants, a reduction of equations (3.2), (3.3) is in order. To that end, we switch to a linearly independent combination $w_1(x)$ and $w_2(x)$ of $Y_1(x)$ and $Y_2(x)$ with the properties $w_1(x_1) = 1$ and $w_2(x_1) = 0$. For example, the definitions

$$w_1(x) = \frac{1}{2}\left\{\frac{Y_1(x)}{Y_1(x_1)} + \frac{Y_2(x)}{Y_2(x_1)}\right\} \tag{3.4}$$

and

$$w_2(x) = \frac{1}{2}\left\{\frac{Y_1(x)}{Y_1(x_1)} - \frac{Y_2(x)}{Y_2(x_1)}\right\} \tag{3.5}$$

accomplish the desired simplification. The general solution is in the form

$$\begin{aligned} W(z) &= CY_1(x_1)\left[w_1(x) + w_2(x)\right] + DY_2(x_1)\left[w_1(x) - w_2(x)\right] \\ &= Fw_1(x) + Gw_2(x). \end{aligned}$$

where $F = CY_1(x_1) + DY_2(x_1)$ and $G = CY_1(x_1) - DY_2(x_1)$.
The sea surface condition reduces to

$$N_s \left(\frac{dx}{dz}\right)_{x=x_0} \left[Fw_1'(x_0) + Gw_2'(x_0)\right] = -\kappa \tag{3.6}$$

and the seabed condition reduces to

$$\sigma_f N_b \left(\frac{dx}{dz}\right)_{x=x_1} \left[Fw_1'(x_1) + Gw_2'(x_1)\right] = -r_0 F\sqrt{F \cdot \bar{F}} \tag{3.7}$$

4 The solution

There is an advantage to expressing F in polar form,

$$F = re^{i\theta}$$

and to using this representation of throughout the development. In terms of transformed boundary conditions the equations are

$$N_s \left(\frac{dx}{dz}\right)_{x=x_0} \left[w_1'(x_0)\, re^{i\theta} + w_2'(x_0)\, G\right] = -\kappa \tag{4.1}$$

$$\sigma_f N_b \left(\frac{dx}{dz}\right)_{x=x_1} \left[w_1'(x_1)\, re^{i\theta} + w_2'(x_1)\, G\right] = r_0 r^2 e^{i\theta} \tag{4.2}$$

It is easy to show that, since $w_2(x_1) = 0$ by design, $w_2'(x_1)$ must be nonzero; otherwise, there is no solution. Then, multiplying each of (4.1) and (4.2) by $e^{-i\theta}$ and eliminating $Ge^{-i\theta}$ we obtain the key relation

$$\frac{r_0 r^2}{\sigma_f} + rA = Be^{-i\theta} \quad \text{or} \quad r\left(\frac{r_0 r}{\sigma_f} + A\right) = Be^{-i\theta} \tag{4.3}$$

where

$$A = N_b \left(\frac{dx}{dz}\right)_{x=x_1} \left[w_1'(x_1) - \frac{w_1'(x_0)\, w_2'(x_1)}{w_2'(x_0)}\right] \tag{4.4}$$

and

$$B = \kappa \frac{N_b \left(\frac{dx}{dz}\right)_{x=x_1}}{N_s \left(\frac{dx}{dz}\right)_{x=x_0}} \left[\frac{w_2'(x_1)}{w_2'(x_0)}\right] \tag{4.5}$$

Next, we eliminate θ by multiplying by the complex conjugate of equation (4.2) to obtain a quartic polynomial in r.
Since $r_0 r^2 + \sigma_f A r = \sigma_f B e^{-i\theta}$ and conjugate $r_0 r^2 + \sigma_f \overline{A} r = \sigma_f \overline{B} e^{i\theta}$, we get

$$\left(r_0 r^2 + \sigma_f A r\right)\left(r_0 r^2 + \sigma_f \overline{A} r\right) = \sigma_f B e^{-i\theta} \sigma_f \overline{B} e^{i\theta}$$

$$r^2 \left(r_0^2 r^2 + \sigma_f r_0 r \left(A + \overline{A}\right) + \sigma_f^2 A\overline{A}\right) - \sigma_f^2 \left(B\overline{B}\right) = 0 \tag{4.6}$$

The smallest positive real root of equation (4.6) is the amplitude of With established, is determined from a modified form of equation (4.5). For F in polar form

$$F = re^{i\theta}$$

and

$$r\left(\frac{r_0 r}{\sigma_f} + A\right) = Be^{-i\theta} \Rightarrow e^{i\theta} = \frac{B}{r \;\; \frac{r_0 r}{\sigma_f} + A}$$

$$F = \frac{B}{\left(\frac{r_0 r}{\sigma_f} + A\right)} \tag{4.7}$$

and therefore either equation (3.7) or (4.1) provides in terms of From (4.1)

$$\sigma_f N_b \left(\tfrac{dx}{dz}\right)_{x=x_1} \left[w_1'(x_1)\, re^{i\theta} + w_2'(x_1)\, G\right] = -r_0 r^2 e^{i\theta}$$

$$G = -F\frac{1}{w_2'(x_1)}\left[\frac{r_0 r}{\sigma_f N_b \left(\tfrac{dx}{dz}\right)_{x=x_1}} + w_1'(x_1)\right] \tag{4.8}$$

When the direction factor is restored, the solution W of equation (2.6) is assembled as

$$\begin{aligned} W(z) &= (Fw_1(x) + Gw_2(x))\, e^{i\left(\frac{\pi}{2}-\chi\right)} \\ &= \left(F\frac{1}{2}\left\{\frac{Y_1(x)}{Y_1(x_1)} + \frac{Y_2(x)}{Y_2(x_1)}\right\} + G\frac{1}{2}\left\{\frac{Y_1(x)}{Y_1(x_1)} - \frac{Y_2(x)}{Y_2(x_1)}\right\}\right) e^{i\left(\frac{\pi}{2}-\chi\right)} \\ &= \frac{1}{2}\left\{\frac{Y_1(x)}{Y_1(x_1)}(F+G) + \frac{Y_2(x)}{Y_2(x_1)}(F-G)\right\} e^{i\left(\frac{\pi}{2}-\chi\right)} \end{aligned} \tag{4.9}$$

Where F is in equation (4.7) and G is in equation (4.8).

5 Examples

The functional representations of eddy viscosity used in this study were $N_v(z) = 1$ and $N_v(z) = (N_s + kz)(1-z) + N_b z$

5.1 Constant eddy viscosity

$$N_v(z) = N_s = N_b = 1$$

The extension of Ekmans problem from the classical bottom no-slip to quadratic stress at the seabed serves as a simple illustration of the method. With $0 < z < 1$ and constant eddy viscosity $N_v(z) = 1$, the governing equation (2.6) becomes

$$\frac{d^2W(z)}{dz^2} - w^2 W(z) = 0$$

where $w = \kappa(1+i)$. Then the transformation $x = w(1-z)$, $z = 0 \Rightarrow x_0$, $z = 1 \Rightarrow x_1$, $x_0 = w(1-0) = w$, $x_1 = w(1-1) = 0$, $\left(\tfrac{dx}{dz}\right)_{x=x_1} = -w$, $\left(\tfrac{dx}{dz}\right)_{x=x_0} = w$. Given $w_1(x) = \cosh(x)$ and $w_2(x) = \sinh(x)$ it is obvious that the properties $w_1(x_1) = 1$ and $w_2(x_1) = 0$ are satisfied. Equations (4.4) and (4.5) provide the constants A and B for substitution into equation (4.3). From equation (4.4)

$$\begin{aligned} A &= N_b\left(\frac{dx}{dz}\right)_{x=x_1}\left[w_1'(x_1) - \frac{w_1'(x_0)\, w_2'(x_1)}{w_2'(x_0)}\right] \\ &= w\tanh(w) \end{aligned}$$

From equation (4.5) $w_2'(x_1) = \cosh(1) = 1$, $w_2'(x_0) = \cosh(w)$

$$\begin{aligned} B &= \kappa \frac{N_b}{N_s} \frac{\left(\frac{dx}{dz}\right)_{x=x_1}}{\left(\frac{dx}{dz}\right)_{x=x_0}} \left[\frac{w_2'(x_1)}{w_2'(x_0)}\right] \\ &= \kappa \frac{-w}{w} \left[\frac{1}{\cosh(w)}\right] \\ &= \kappa \operatorname{sech}(w) \end{aligned}$$

After the quartic obtained from equation (4.3) is solved for r the coefficients F and G follow from (4.4) and (4.5) :

$$F = \frac{\kappa \operatorname{sech}(w)}{\frac{r}{\sigma_f} + w\tanh(w)} \quad \text{and} \quad G = F\left(\frac{r}{\sigma_f w}\right).$$

The solution of the original problem then reads

$$\begin{aligned} W(z) &= (F w_1(x) + G w_2(x))\, e^{i\left(\frac{\pi}{2}-\chi\right)} \\ &= F\left(w_1(x) + \frac{r}{\sigma_f w} w_2(x)\right) e^{i\left(\frac{\pi}{2}-\chi\right)} \\ &= \frac{\kappa \operatorname{sech}(w)}{\left(\frac{r}{\sigma_f} + w\tanh(w)\right)} \left[\cosh(w(1-z)) + \frac{r}{\sigma_f w}\sinh(w(1-z))\right] e^{i\left(\frac{\pi}{2}-\chi\right)} \end{aligned}$$

where $w = \kappa(1+i)$.

5.2 Parabolic Profiles

The quadratic form selected here is more general than the forms used by Fjeldsted[1929], John[1966] and Noye and Stevens[1987] :

$$N_v(z) = (N_s + kz)(1-z) + N_b z$$

Let $N_s = 1 + ka$ and $k - N_s + N_b = kb$

$$N_v(z) = 1 + k\left(a + bz - z^2\right)$$

Solutions were studied for parabolic profiles with critical points at 0, $\frac{1}{3}$ and $\frac{1}{2}$. Although the critical point of choice is $\frac{1}{3}$, when $N_b = 1$, and N_s is larger than 1 (we use the example $N_s = 2$). The analysis is the same for each profile, and the general form of the final solution is similar for each instance. A suitable transformation is

$$z = \frac{1}{2}(kx + b), \quad K = \sqrt{\frac{4}{k} + (b^2 + 4a)}$$

$$N_v(z) = 2 + (k-1)z - kz^2$$

$$N_{\upsilon}'(z) = k - 1 - 2kz \ , \ z = \frac{k-1}{2k}$$

At $z = \frac{1}{3}$, $k = 3$ we obtain from $N_{\upsilon}(z) = 2 + 2z - 3z^2 = 1 + 3\left(\frac{1}{3} + \frac{2}{3}z - z^2\right)$ that $\therefore \ a = \frac{2}{3}, b = \frac{1}{3}$ and $k = 3$. Then , from $K = \sqrt{\frac{28}{9}}$, $N_{\upsilon}(z) = \frac{7}{3} - \frac{7}{3}x^2$ and from

$$\frac{d}{dz}\left(N_{\upsilon}(z)\frac{d}{dz}W(z)\right) - i\frac{6K^2}{k}W(z) = 0$$

$$3\left[\left(1-x^2\right)\frac{d^2Y(x)}{dx} + 2x\frac{dY(x)}{dx}\right] - i\frac{6K^2}{k}Y(x) = 0$$

$$\therefore \ \left(1-x^2\right)\frac{d^2Y(x)}{dx} + 2x\frac{dY(x)}{dx} - i\frac{2K^2}{k}Y(x) = 0.$$

The linearly independent solutions of the differential equation are the Legende functions of the first and second kind of order zero and complex degree υ, where

$$\upsilon = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - i\frac{8\kappa^2}{k}}$$

and

$$a_1 = -\frac{1}{2}\upsilon \quad = \frac{1}{4} - \frac{1}{4}\sqrt{1 - i\frac{8\kappa^2}{k}} \quad , \quad b_1 = \frac{1}{2} + \frac{1}{2}\upsilon \quad = \frac{1}{4} + \frac{1}{4}\sqrt{1 - i\frac{8\kappa^2}{k}}$$

$$a_2 = \frac{1}{2} - \frac{1}{2}\upsilon \quad = \frac{3}{4} - \frac{1}{4}\sqrt{1 - i\frac{8\kappa^2}{k}} \quad , \quad b_2 = 1 + \frac{1}{2}\upsilon \quad = \frac{3}{4} + \frac{1}{4}\sqrt{1 - i\frac{8\kappa^2}{k}}$$

The Legende functions can be computed as combinations of hypergeometric functions,

$$P_{\upsilon}(x) = \sqrt{\pi}\left[\frac{F\left(a_1, b_1; \frac{1}{2}; x^2\right)}{\Gamma(a_2)\Gamma(b_2)} - 2x\frac{F\left(a_2, b_2; \frac{3}{2}; x^2\right)}{\Gamma(a_1)\Gamma(b_1)}\right].$$

where

$$F\left(a_1, b_1; \tfrac{1}{2}; x^2\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(a_1)\Gamma(b_1)}\sum_{n=0}^{\infty}\frac{\Gamma(n+a_1)\Gamma(n+b_1)}{\Gamma\left(n+\frac{1}{2}\right)n!}x^{2n}$$

$$F\left(a_2, b_2; \tfrac{3}{2}; x^2\right) = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(a_2)\Gamma(b_2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+a_2)\Gamma(n+b_2)}{\Gamma\left(n+\frac{3}{2}\right)n!}x^{2n}$$

and

$$Q_{\upsilon}(x) = \pi^{\frac{3}{2}}\left[-\frac{1}{2}\tan\left(\frac{\pi\upsilon}{2}\right)\frac{F\left(a_1, b_1; \frac{1}{2}; x^2\right)}{\Gamma(a_2)\Gamma(b_2)} + \cot\left(\frac{\pi\upsilon}{2}\right)\frac{F\left(a_2, b_2; \frac{3}{2}; x^2\right)}{\Gamma(a_1)\Gamma(b_1)}\right].$$

The constructions of $w_1(x)$ and $w_2(x)$ are

$$w_1(x) = \frac{1}{2}\left[\frac{P_{\upsilon}(x)}{P_{\upsilon}(x_1)} + \frac{Q_{\upsilon}(x)}{Q_{\upsilon}(x_1)}\right]$$

and

$$w_2(x) = \frac{1}{2}\left[\frac{P_\upsilon(x)}{P_\upsilon(x_1)} - \frac{Q_\upsilon(x)}{Q_\upsilon(x_1)}\right]$$

The complex velocity as a function of $x = \frac{2z-b}{K}$ is

$$\begin{aligned} Y(x) &= (Fw_1(x) + Gw_2(x))\, e^{i\left(\frac{\pi}{2}-\chi\right)} \\ &= \frac{1}{2}\left[\frac{P_\upsilon(x)}{P_\upsilon(x_1)}(F+G) - \frac{Q_\upsilon(x)}{Q_\upsilon(x_1)}(F-G)\right] e^{i\left(\frac{\pi}{2}-\chi\right)} \\ \therefore\ W(z) &= \frac{1}{2}\left[\frac{P_\upsilon\left(\frac{2z-b}{K}\right)}{P_\upsilon\left(\frac{2-b}{K}\right)}(F+G) - \frac{Q_\upsilon\left(\frac{2z-b}{K}\right)}{Q_\upsilon\left(\frac{2-b}{K}\right)}(F-G)\right] e^{i\left(\frac{\pi}{2}-\chi\right)}. \end{aligned}$$

where F is in equation (4.7) and G is in equation (4.8).

6 Conclusion

In this paper, it may be worth emphasizing that the present development was designed to handle problems in fairly shallow water with nonlinear bottom stress together with depth-dependent eddy viscosity and rotation effects. The use of a linearized bottom stress is well established. The linearized conditions can be estimated following the illustrations in this work.

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