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Complete Blow-up for a Semilinear Parabolic Equation

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Abstract: Let $T \leqslant \infty$, a and x_0 be constants with $a > 0$ and $0 < x_0 < a$. We establish the unique solution u for the following semilinear parabolic initial-boundary value problem:

$$\begin{aligned} u_t(x,t) - u_{xx}(x,t) &= f(u(x_0,t)) \quad \text{for } 0 < x < a,\ 0 < t < T, \\ u(x,0) &= \phi(x) \geqslant 0 \quad \text{for } 0 \leqslant x \leqslant a, \\ u(0,t) = u_x(a,t) &= 0 \quad \text{for } 0 < t < T, \end{aligned}$$

where f and ϕ are given functions. We also show that under certain conditions, u blows up in a finite time, and the set of the blow-up points is the entire interval $[0,a]$.

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1 Introduction

Let $T \leqslant \infty$, and a and x_0 be constants with $a > 0$ and $0 < x_0 < a$. Let $D = (0,a), \Omega = D \times (0,T)$, $\bar{D}, \bar{\Omega}$ be their respective closures, and $Lu = u_t - u_{xx}$. Consider the following semilinear parabolic initial-boundary value problem:

$$\begin{aligned} Lu(x,t) &= f(u(x_0,t)) \quad \text{in } \Omega, \\ u(x,0) &= \phi(x) \geqslant 0 \quad \text{on } \bar{D}, \\ u(0,t) = u_x(a,t) &= 0 \quad \text{for } 0 < t < T, \end{aligned} \tag{1.1}$$

where $f \in C^2([0,\infty)), f(0) \geqslant 0, f'(s) > 0$ and $f''(s) \geqslant 0$ for $s > 0$, $\int_{z_0}^{\infty} (f(s))^{-1}\, ds < \infty$ for some $z_0 > 0$, and $\phi(x)$ is nontrivial, nonnegative and continuous such that $\phi(0) = \phi'(a) = 0$ and

$$\phi''(x) + f^2(\phi(x)) \geqslant 0 \quad \text{in } D. \tag{1.2}$$

A solution u is said to blow up at the point $(\tilde{x},T)$ if there exists a sequence $\{(x_n,t_n)\}$ such that $\lim\limits_{n\to\infty} u(x_n,t_n) \to \infty$ as $(x_n,t_n) \to (\tilde{x},T)$. Furthermore, if u blows up at every point $x \in \bar{D}$ at T, then the complete blow-up occurs. We note that the condition (1.2) is used to show that before u blows up, u is a nondecreasing function of t.

2 Existence and Uniqueness

Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (1.1) is determined by the following system: for x and ξ in D and t and τ in $(0,T)$,

$$\begin{aligned} LG(x,t;\xi,\tau) &= \delta(x-\xi)\delta(t-\tau), \\ G(x,t;\xi,\tau) &= 0 \quad \text{for } t<\tau, \\ G(0,t;\xi,\tau) &= G_x(a,t;\xi,\tau)=0. \end{aligned} \tag{2.3}$$

By the method of eigenfunction expansion,

$$G(x,t;\xi,\tau)=\sum_{k=1}^{\infty} g_k(x)g_k(\xi)\exp(-\lambda_k(t-\tau)) \quad \text{for } t>\tau,$$

where $\lambda_k = ((2k-1)\pi/(2a))^2$ and $g_k(x)=\sqrt{2/a}\sin(\sqrt{\lambda_k}x)$, $k=1,2,3,\ldots$ are the eigenvalues and the eigenfunctions of the Sturm-Liouville problem: $g''(x)+\lambda g(x)=0$, $g(0)=g'(a)=0$. Note that $\left|\sum_{k=1}^{\infty} g_k(x)g_k(\xi)\exp[-\lambda_k(t-\tau)]\right| \leqslant (2/a)\sum_{k=1}^{\infty}\exp[-\lambda_k(t-\tau)]$ for $t>\tau$ which converges uniformly. Thus, the Green's function exists.

Consider the adjoint operator L^* given by $L^*u=-u_t-u_{xx}$. Using Green's second identity, we obtain an integral equation equivalent to the problem (1.1):

$$u(x,t)=\int_0^t\int_0^a G(x,t;\xi,\tau)f(u(x_0,\tau))\,d\xi\,d\tau+\int_0^a G(x,t;\xi,0)\phi(\xi)\,d\xi. \tag{2.4}$$

Similar to Lemma 4 of Chan and Wong [3] and Lemma 2.2(d) of Chan and Tian [2], we have the following properties of the Green's function.

Lemma 1 *G is positive in the set $D_1=\{(x,t;\xi,\tau): x$ and ξ are in $D,\ 0\leqslant\tau<t\leqslant T\}$.*

Proof. Suppose that there exists a point $(x_1,t_1;\xi_1,\tau_1)$ in D_1 such that $G<0$. Since G converges uniformly in D_1, G is continuous and we may assume $\tau>0$. Hence, there exists $\varepsilon>0$ such that $G<0$ in

$$W=(x_1-\varepsilon,x_1+\varepsilon)\times(t_1-\varepsilon,t_1+\varepsilon)\times(\xi_1-\varepsilon,\xi_1+\varepsilon)\times(\tau_1-\varepsilon,\tau_1+\varepsilon)\subset D_1.$$

Define $h(x,t)=\exp\left\{-1/\left[\left(\varepsilon^2-(x-x_1)^2\right)\left(\varepsilon^2-(t-t_1)^2\right)\right]\right\}$ in Ω_ε and $h(x,t)=0$ outside Ω_ε, where $\Omega_\varepsilon=(\xi_1-\varepsilon,\xi_1+\varepsilon)\times(\tau_1-\varepsilon,\tau_1+\varepsilon)$. The solution of the problem $Lu=h$ in $D\times(0,\alpha)$, $\alpha>\tau_1+\varepsilon$ with u satisfying zero initial condition and the boundary conditions $u(0)=u'(a)=0$ is given by

$$u(x,t)=\int_{\tau_1-\varepsilon}^{\tau_1+\varepsilon}\int_{\xi_1-\varepsilon}^{\xi_1+\varepsilon} G(x,t;\xi,\tau)h(\xi,\tau)\,d\xi\,d\tau.$$

Since $G < 0$ in W, $h > 0$ in Ω_ε, it follows that $u < 0$ in Ω_ε. On the other hand, $h \geqslant 0$ in $D \times (0, \alpha)$ implies $u \geqslant 0$ by the weak maximum principle and Hopf's Lemma. This leads to a contradiction. Therefore, $G \geqslant 0$ in D_1.

We shall show that $G \neq 0$ in D_1. Suppose that there exists a point $(x_2, t_2; \xi_2, \tau_2)$ in D_1 such that $G = 0$. Using the strong maximum principle, we have $G = 0$ in $D_1 \cap \{(x, t; \xi_2, \tau_2) : 0 < x < a,\ t \leqslant t_2\}$. On the other hand,

$$G(\xi_2, t_2; \xi_2, \tau_2) = \frac{2}{a} \sum_{k=1}^{\infty} \sin^2(\sqrt{\lambda_k}\xi_2) \exp[-\lambda_k(t_2 - \tau_2)] > 0.$$

We have a contradiction. This shows that G is positive in D_1. □

Lemma 2 *For any function* $\gamma \in C([0, T])$, $\int_0^t \int_0^a G(x, t; \xi, \tau)\gamma(\tau)\, d\xi\, d\tau$ *is continuous on* Ω.

Proof. For $x \in \bar{D}$, $\tau \in [0, t - \varepsilon]$, and $0 < \varepsilon < t$,

$$\sum_{k=1}^{\infty} g_k(x) g_k(\xi) \exp\left[-\lambda_k(t - \tau)\right] \gamma(\tau) \leqslant \frac{2}{a} \left(\max_{0 \leqslant \tau \leqslant T} \gamma(\tau) \right) \sum_{k=1}^{\infty} \exp\left[-\lambda_k(t - \tau)\right]$$

converges uniformly. It follows that

$$\begin{aligned} \int_0^{t-\varepsilon} \int_0^a G(x, t; \xi, \tau)\gamma(\tau)\, d\xi\, d\tau &= \sum_{k=1}^{\infty} \int_0^{t-\varepsilon} \int_0^a g_k(x) g_k(\xi) \exp\left[-\lambda_k(t - \tau)\right] \gamma(\tau)\, d\xi\, d\tau \\ &\leqslant \frac{2}{a} \left(\max_{0 \leqslant \tau \leqslant T} \gamma(\tau) \right) \sum_{k=1}^{\infty} \int_0^{t-\varepsilon} \int_0^a \exp\left[-\lambda_k(t - \tau)\right]\, d\xi d\tau \\ &\leqslant \frac{2}{a} \left(\max_{0 \leqslant \tau \leqslant T} \gamma(\tau) \right) \sum_{k=1}^{\infty} \lambda_k^{-1} \end{aligned}$$

converges uniformly with respect to x, t and ε. Since the uniform convergence also holds for $\varepsilon = 0$, $\sum_{k=1}^{\infty} \int_0^{t-\varepsilon} \int_0^1 g_k(x) g_k(\xi) \exp\left[-\lambda_k(t - \tau)\right] \gamma(\tau)\, d\xi\, d\tau$ is a continuous function of x, t and $\varepsilon \geqslant 0$. Therefore,

$$\int_0^t \int_0^a G(x, t; \xi, \tau)\gamma(\tau)\, d\xi\, d\tau = \lim_{\varepsilon \to 0} \sum_{k=1}^{\infty} \int_0^{t-\varepsilon} \int_0^a g_k(x) g_k(\xi) \exp\left[-\lambda_k(t - \tau)\right] \gamma(\tau)\, d\xi\, d\tau$$

is a continuous function of x and t. □

Based on Theorem 2.4 of Chan and Tian [2], we prove the following theorem.

Theorem 3 *There exists some* t_0 *such that for* $0 \leqslant t \leqslant t_0$, *the integral equation (2.4) has the unique continuous solution* $u \geqslant \phi(x)$ *and* u *is a nondecreasing function of* t. *Let* t_b *be the supremum of the interval for which the integral equation (2.4) has the unique continuous solution. If* t_b *is finite, then* $u(x_0, t)$ *is unbounded in* $[0, t_b)$.

Proof. Construct a sequence $\{u_n\}$ in Ω by $u_0(x,t) = \phi(x)$, and for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} Lu_{n+1}(x,t) &= f(u_n(x_0,t)) \quad \text{in } \Omega, \\ u_{n+1}(x,0) &= \phi(x) \quad \text{on } \bar{D}, \\ u_{n+1}(0,t) &= (u_{n+1})_x(a,t) = 0 \quad \text{for } 0 < t < T. \end{aligned}$$

We have

$$\begin{aligned} L(u_1 - u_0)(x,t) &= f(u_0(x_0,t)) + \phi''(x) \geqslant 0 \quad \text{in } \Omega, \\ (u_1 - u_0)(x,0) &= 0 \quad \text{on } \bar{D}, \\ (u_1 - u_0)(0,t) &= (u_1 - u_0)_x(a,t) = 0 \quad \text{for } 0 < t < T. \end{aligned}$$

It follows from (2.4) and Lemma 1 that $u_1 \geqslant u_0$ in Ω. Assume that for a positive integer m

$$\phi \leqslant u_1 \leqslant u_2 \leqslant \ldots \leqslant u_{m-1} \leqslant u_m \quad \text{in } \Omega.$$

Since f is increasing and $u_m \geqslant u_{m-1}$, we have

$$\begin{aligned} L(u_{m+1} - u_m)(x,t) &= f(u_m(x_0,t)) - f(u_{m-1}(x_0,t)) \geqslant 0 \quad \text{in } \Omega, \\ (u_{m+1} - u_m)(x,0) &= 0 \quad \text{on } \bar{D}, \\ (u_{m+1} - u_m)(0,t) &= (u_{m+1} - u_m)_x(a,t) = 0 \quad \text{for } 0 < t < T. \end{aligned}$$

By (2.4) and Lemma 1, $u_{m+1} \geqslant u_m$. By the principle of mathematical induction,

$$\phi \leqslant u_1 \leqslant u_2 \leqslant \ldots \leqslant u_{n-1} \leqslant u_n \quad \text{in } \Omega \text{ for all positive integer } n. \tag{2.5}$$

We shall also show that the sequence $\{u_n\}$ is a nondecreasing function of t. Let $w_n(x,t) = u_n(x,t+h) - u_n(x,t)$ for $n = 0, 1, 2, \ldots$, where $0 < h < T - t$. Then $w_0(x,t) = 0$, and

$$\begin{aligned} Lw_1(x,t) &= f(u_0(x_0,t+h)) - f(u_0(x_0,t)) = 0 \quad \text{in } D \times (0, T-h), \\ w_1(x,0) &= u_1(x,h) - \phi(x) \geqslant 0 \quad \text{on } \bar{D}, \\ w_1(0,t) &= (w_1)_x(a,t) = 0 \quad \text{for } 0 < t < T - h. \end{aligned}$$

By (2.4) and Lemma 1, $w_1 \geqslant 0$. Let us assume that $w_m \geqslant 0$ for a positive integer m. Using the Mean Value Theorem, we have

$$Lw_{m+1} = f(u_m(x_0,t+h)) - f(u_m(x_0,t)) = f'(\zeta_m)w_m(x,t) \geqslant 0$$

for some ζ_m between $u_m(x_0,t+h)$ and $u_m(x_0,t)$. Also,

$$\begin{aligned} w_{m+1}(x,0) &\geqslant 0 \quad \text{on } \bar{D} \\ w_{m+1}(0,t) &= (w_{m+1})_x(a,t) = 0 \quad \text{for } 0 < t < T - h. \end{aligned}$$

It follows that $w_{m+1} \geqslant 0$ in Ω. By the principle of mathematical induction, $w_n \geqslant 0$ for all positive integer n. This shows that u_n is a nondecreasing function of t.

Consider the problem:

$$Lv(x,t) = 0 \quad \text{in } \Omega,$$
$$v(x,0) = \phi(x) \geqslant 0 \quad \text{on } \bar{D},$$
$$v(0,t) = v_x(a,t) = 0 \quad \text{for } 0 < t < T.$$

By (2.4) and Lemma 1, $v \geqslant 0$ in Ω. By the strong maximum principle, v attains its maximum somewhere on $(D \times \{0\}) \cup (\{a\} \times (0,T))$. By Hopf's Lemma, v attains its maximum $k_0 = \max_{\bar{D}} \phi(x)$ on $D \times \{0\}$.

For a given positive constant $M > k_0$, consider

$$u_n(x,t) = \int_0^t \int_0^a G(x,t;\xi,\tau) f(u_{n-1}(x_0,\tau))\, d\xi\, d\tau + \int_0^a G(x,t;\xi,0)\phi(\xi)\, d\xi. \quad (2.6)$$

By Lemma 2, as $t \to 0$, we see that

$$\lim_{t\to 0} u_n(x,t) = \int_0^a \lim_{t\to 0} G(x,t;\xi,0)\phi(\xi)\, d\xi = \phi(x) < M.$$

This shows that there exists t_2 such that $u_n(x,t) \leqslant M$ for $0 \leqslant t \leqslant t_2$. In fact, if we choose t_2 satisfying

$$f(M) \int_0^{t_2} \int_0^a G(x,t_2;\xi,\tau)\, d\xi\, d\tau + \int_0^a G(x,t_2;\xi,0)\phi(\xi)\, d\xi \leqslant M,$$

then the inequality holds.

Let u denote $\lim_{n\to\infty} u_n$. From (2.6) and the Monotone Convergence Theorem, we have (2.4) for $0 \leqslant t \leqslant t_2$.

We proceed to show that $\{u_n\}$ converges uniformly to u for $0 \leqslant t \leqslant t_2$. Consider

$$u_{n+1}(x,t) - u_n(x,t) = \int_0^t \int_0^a G(x,t;\xi,\tau) \left[f(u_n(x_0,\tau)) - f(u_{n-1}(x_0,\tau))\right] d\xi\, d\tau.$$

Let $S_n = \max_{\bar{D}\times[0,t_2]} (u_n - u_{n-1})$. Using the Mean Value Theorem and $f''(s) > 0$ for $s > 0$,

$$f(u_n(x_0,\tau)) - f(u_{n-1}(x_0,\tau)) \leqslant f'(M) S_n.$$

Thus,

$$S_{n+1} \leqslant \frac{2}{a} f'(M) S_n \sum_{k=1}^{\infty} \int_0^t \int_0^a \exp\left[-\lambda_k(t-\tau)\right] d\xi\, d\tau = 2f'(M) S_n \sum_{k=1}^{\infty} \lambda_k^{-1}[1-\exp(-\lambda_k t)].$$

Since $\sum_{k=1}^{\infty} \lambda_k^{-1}[1-\exp(-\lambda_k t)]$ converges uniformly, we have $\lim_{t\to 0} \sum_{k=1}^{\infty} \lambda_k^{-1}(1-\exp(-\lambda_k t)) = 0$. Hence, there exists some $\sigma_1 > 0$ such that

$$2f'(M) \sum_{k=1}^{\infty} \lambda_k^{-1}(1 - \exp(-\lambda_k t)) < 1 \quad \text{for } t \in [0,\sigma_1].$$

Thus, $S_{n+1} < S_n$ and the sequence $\{u_n\}$ converges uniformly to u for $0 \leqslant t \leqslant \sigma_1$.

Similarly for $\sigma_1 \leqslant t \leqslant t_2$, we replace $\phi(\xi)$ in (2.4) by $u(\xi, \sigma_1)$ to obtain

$$u_n(x,t) = \int_{\sigma_1}^{t} \int_0^a G(x,t;\xi,\tau) f(u_{n-1}(x_0,\tau))\, d\xi\, d\tau + \int_0^a G(x,t;\xi,0) u(\xi,\sigma_1)\, d\xi$$

and

$$\begin{aligned} S_{n+1} &\leqslant \frac{2}{a} f'(M) S_n \sum_{k=1}^{\infty} \int_{\sigma_1}^{t} \int_0^a \exp\left[-\lambda_k (t-\tau)\right] d\xi\, d\tau \\ &= 2f'(M) S_n \sum_{k=1}^{\infty} \lambda_k^{-1} (1 - \exp(-\lambda_k (t - \sigma_1))). \end{aligned}$$

Thus, there exists $\sigma_2 = \min\{\sigma_1, t_2 - \sigma_1\} > 0$ such that

$$2f'(M) \sum_{k=1}^{\infty} \lambda_k^{-1} (1 - \exp(-\lambda_k (t-\sigma_1))) < 1 \quad \text{for } t \in [\sigma_1, \min\{2\sigma_1, t_2\}].$$

Hence, the sequence $\{u_n\}$ converges uniformly to u for $t \in [\sigma_1, \min\{2\sigma_1, t_2\}]$.

By proceeding in this way, the sequence $\{u_n\}$ converges uniformly to u for $0 \leqslant t \leqslant t_2$. Therefore, the integral equation (2.4) has a continuous solution u for $0 \leqslant t \leqslant t_2$.

To show that the solution is unique, let us suppose that the integral equation (2.4) has two distinct solutions u and $\tilde{u}$ on the interval $[0, t_2]$. Also, let $\Phi = \max_{\bar{D} \times [0,t_2]} |u - \tilde{u}| > 0$. We have

$$u(x,t) - \tilde{u}(x,t) = \int_0^t \int_0^a G(x,t;\xi,\tau) \left[f(u(x_0,\tau)) - f(\tilde{u}(x_0,\tau))\right] d\xi\, d\tau.$$

Then

$$\Phi \leqslant 2f'(M) \left[\sum_{k=1}^{\infty} \lambda_k^{-1} (1 - \exp(-\lambda_k t))\right] \Phi \quad \text{for } t \in \ [0, \sigma_1],$$

which implies that

$$2f'(M) \left[\sum_{k=1}^{\infty} \lambda_k^{-1} (1 - \exp(-\lambda_k t))\right] \geqslant 1 \quad \text{for } t \in \ [0, \sigma_1].$$

We have a contradiction. Hence, the solution is unique for $0 \leqslant t \leqslant \sigma_1$.

We can show in a similar fashion that the solution u_n is unique for $\sigma_1 \leqslant t \leqslant \min\{2\sigma_1, t_2\}$. Continuing in this way, the integral (2.4) has the unique continuous solution u for $0 \leqslant t \leqslant t_2$.

Let t_b be the supremum of the interval for which the integral equation (2.4) has the unique continuous solution u. We would like to show that if t_b is finite, then $u(x_0,t)$ is unbounded in $[0,t_b)$. Suppose that $u(x_0,t)$ is bounded in $[0,t_b)$. Consider (2.4) for $t \in [t_b,T)$ with the initial condition $u(x,0)$ replaced by $u(x,t_b)$:

$$u(x_0,t) = \int_{t_b}^{t}\int_0^a G(x_0,t,\xi,\tau)f(u(x_0,\tau))\,d\xi\,d\tau + \int_0^a G(x_0,t,\xi,t_b)u(\xi,t_b)\,d\xi.$$

For any positive constant $N > u(x_0,t_b)$, an argument as before shows that there exists t_3 such that the integral equation (2.4) has the unique continuous solution u on $[t_b,t_3]$. This contradicts the definition of t_b. Hence, if t_b is finite, then $u(x_0,t)$ is unbounded in $[0,t_b)$. Moreover, u is a nondecreasing function of t since u_n is a nondecreasing function of t. □

3 A Sufficient Condition for Blow-Up in a Finite Time

Based on Lemma 1 of Chan and Yang [4], we prove the following lemma.

Lemma 4 *Let u be a solution of the following problem:*

$$\begin{aligned} &Lu(x,t) = b(x,t)u(x_0,t) \quad \text{in } \Omega,\\ &u(x,0) \geqslant 0 \quad \text{on } \bar{D},\\ &u(0,t) = u_x(a,t) = 0 \quad \text{for } 0<t<T, \end{aligned}$$

where $b(x,t)$ is nonnegative and bounded, then $u(x,t) \geqslant 0$ in Ω.

Proof. If $b(x,t) \equiv 0$, the strong maximum principle can be applied to obtain the conclusion immediately. If $b(x,t)$ is nontrivial, let η be a positive constant, and

$$V(x,t) = u(x,t) + \eta(1+\sqrt{x})e^{ct},$$

where c is a constant. Then $V(x,t) > 0$ in $(\{0\}\times(0,T)) \cup ([0,a]\times\{0\})$, and

$$\begin{aligned} LV(x,t) - b(x,t)V(x_0,t) &\geqslant L[\eta(1+\sqrt{x})e^{ct}] - b(x,t)\eta(1+\sqrt{x_0})e^{ct}\\ &= \eta e^{ct}\left[c(1+\sqrt{x}) - b(x,t)(1+\sqrt{x_0}) + \frac{1}{4x\sqrt{x}}\right]. \end{aligned}$$

Choose a constant $c \geqslant (1+\sqrt{x_0})\max_{x\in\Omega} b(x,t)$, then $LV(x,t) - b(x,t)V(x_0,t) > 0$ in Ω. Suppose $V(x,t) < 0$ somewhere in Ω, we let $\bar{t} = \inf\{t : V(x,t) \leqslant 0 \text{ for some } x \in D\}$. Since $V(x,0) > 0$ and $V_x(a,t) = \eta e^{ct}/(2\sqrt{a}) > 0$, we have $0 < \bar{t} < T$ and there exists $x_1 \in D$ such that $V(x_1,\bar{t}) = 0$ and $V_t(x_1,\bar{t}) \leqslant 0$. On the other hand, since $V(x,t)$ attains its local minimum at $(x_1,\bar{t})$, we have $V_{xx}(x_1,\bar{t}) \geqslant 0$. Since $V(x_0,\bar{t}) \geqslant 0$, we have

$$0 < LV(x_1,\bar{t}) - b(x_1,\bar{t})V(x_0,\bar{t}) \leqslant V_t(x_1,\bar{t}) \leqslant 0.$$

This contradiction shows that $V(x,t) > 0$ in Ω. It follows that $V(a,t) \geqslant 0$ for $0 \leqslant t < T$. As $\eta \to 0^+$, we conclude that $u(x,t) \geqslant 0$ in Ω. □

Similar to Theorem 8 of Chan and Yang [4], we state a sufficient condition for blow-up in a finite time in the following theorem.

Theorem 5 *If $\phi(x)$ is sufficiently large in a neighborhood of x_0, then u blows up in a finite time.*

Proof. Let us consider the following problem,

$$\begin{aligned} Lv(x,t) &= f(v(x_0,t)) \quad \text{in } (x_0-\delta, x_0) \times (0,T), \\ v(x,0) &= v_0(x) \geqslant 0 \quad \text{on } [x_0-\delta, x_0], \\ v(x_0-\delta,t) &= v_x(x_0,t) = 0 \quad \text{for } 0 < t < T, \end{aligned}$$

where $v_0(x)$ is nondecreasing and $v_0'(x_0) = 0$. If $\lim_{x\to\infty}(f(x)/x) < \infty$, then there exists a positive constant N such that $\lim_{x\to\infty}(f(x)/x) \leqslant N$, which contradicts $\int_{z_0}^{\infty}(f(s))^{-1}\,ds < \infty$. Thus, $\lim_{x\to\infty}(f(x)/x) = \infty$.

Since $\lambda_1 > 0$, there exists a positive constant $k_1 > z_0$ such that

$$\frac{f(x)}{x} \geqslant \max\{2\lambda_1, \frac{2}{\delta^2}\} \quad \text{for } x \geqslant k_1.$$

Therefore, $f(x) > f(x) - \lambda_1 x \geqslant f(x)/2$ for $x \geqslant k_1$, which gives $\int_{k_1}^{\infty}(f(x) - \lambda_1 x)^{-1}\,dx < \infty$. ¿From Samarskii, Galaktionov, Kurdyumov and Mikhailov [8], v blows up in a finite time at $x = x_0$ provided that $v_0(x)$ is large enough. Choose a positive constant $k_2 \geqslant k_1/\delta^2$ big enough such that

$$w_0(x) = k_2[x-(x_0-\delta)][(x_0+\delta)-x] \geqslant v_0(x) \quad \text{in } [x_0-\delta, x_0].$$

Then $w_0(x_0-\delta) = w_0'(x_0) = 0$. Let us consider the following problem:

$$\begin{aligned} Lw(x,t) &= f(w(x_0,t)) \quad \text{in } (x_0-\delta, x_0) \times (0,T), \\ w(x,0) &= w_0(x) \quad \text{on } [x_0-\delta, x_0], \\ w(x_0-\delta,t) &= w_x(x_0,t) = 0 \quad \text{for } 0 < t < T. \end{aligned}$$

By Lemma 4, $w \geqslant v$ in $[x_0-\delta, x_0] \times [0,T)$, and w blows up in a finite time. If we choose $\phi \geqslant w_0$ in $[x_0-\delta, x_0] \times [0,T)$, then $u \geqslant w$. Therefore, u blows up in a finite time, provided that $\phi(x)$ is sufficiently large in some neighborhood of x_0. □

4 Complete Blow-Up

We state additional properties of the Green's function in the following lemma.

Lemma 6 *Given any $x \in D$ and any finite T, there exists positive constants C_1 and C_2 such that*

$$C_1 < \int_0^a G(x,t;\xi,0)\,d\xi < C_2 \quad \textit{for } 0 \leqslant t \leqslant T.$$

Proof. Let us consider the following auxiliary problem:

$$\begin{aligned} Lv(x,t) &= 1 \quad \text{in } \Omega, \\ v(x,0) &= 0 \quad \text{on } \bar{D}, \\ v(0,t) &= v_x(1,t) = 0 \quad \text{for } 0 < t < T, \end{aligned}$$

which has the unique solution v given by

$$v(x,t) = \int_0^t \int_0^a G(x,t-\tau;\xi,0)\,d\xi\,d\tau = \int_0^t \int_0^a G(x,\tau;\xi,0)\,d\xi\,d\tau.$$

It follows that

$$v_t(x,t) = \int_0^a G(x,t;\xi,0)\,d\xi > 0.$$

Since

$$v_t(x,0) = \int_0^a G(x,0;\xi,0)\,d\xi = \int_0^a \sum_{k=1}^{\infty} g_k(x)g_k(\xi)\,d\xi = 1,$$

there exists a positive C_1 such that

$$C_1 < \int_0^a G(x,t;\xi,0)\,d\xi \quad \text{for } 0 \leqslant t \leqslant T.$$

Furthermore, since $v_t(x,t)$ is continuous in $D \times [0,T]$, there exists a positive C_2 such that

$$\int_0^a G(x,t;\xi,0)\,d\xi < C_2 \quad \text{for } 0 \leqslant t \leqslant T.$$

□

We finally state the incidence of complete blow-up in the following theorem.

Theorem 7 *If the solution of the problem (1.1) blows up in a finite time T, then the blow-up set is $\bar{D}$.*

Proof. For any $t < T$,

$$u(x,t) = \int_0^t \int_0^a G(x,t-\tau;\xi,0)f(u(x_0,\tau))\,d\xi\,d\tau + \int_0^a G(x,t;\xi,0)\phi(\xi)\,d\xi.$$

If u blows up in a finite time T, then we know by Theorem 3 that u blows up at least at $x = x_0$. By Lemma 6,

$$\begin{aligned} u(x_0,t) &= \int_0^t \int_0^a G(x_0,\tau;\xi,0)f(u(x_0,t-\tau))\,d\xi\,d\tau + \int_0^a G(x_0,t;\xi,0)\phi(\xi)\,d\xi \\ &\leqslant C_2 \int_0^t f(u(x_0,t-\tau))\,d\tau + C_2 \max_{x\in\bar{D}} \phi(x). \end{aligned}$$

Since $u(x_0,t)\to\infty$ as $t\to T$, we have $\int_0^T f(u(x_0,T-\tau))\,d\tau=\infty$.

On the other hand,

$$u(x,t)\geqslant C_1\int_0^t f(u(x_0,t-\tau))\,d\tau+\int_0^a G(x,t;\xi,0)\phi(\xi)\,d\xi\geqslant C_1\int_0^t f(u(x_0,t-\tau))\,d\tau.$$

As t approaches T^-, it follows from $\int_0^T f(u(x_0,T-\tau))\,d\tau\to\infty$ that $u(x,t)$ tends to infinity. Thus, the blow-up set is D. For $\tilde{x}\in\{0,a\}$, we can always find a sequence $\{(x_n,t_n)\}$ such that $(x_n,t_n)\to(\tilde{x},T)$ and $\lim_{n\to\infty}u(x_n,t_n)\to\infty$. Therefore, the blow-up set is $\bar{D}$. □

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